Steady State Learning and the Code of Hammurabi

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Introduction

"If any one bring an accusation against a man, and the accused go to the river and leap into the river, if he sink in the river his accuser shall take possession of his house. But if the river prove that the accused is not guilty, and he escape unhurt, then he who had brought the accusation shall be put to death, while he who leaped into the river shall take possession of the house that had belonged to his accuser."
[2nd law of Hammurabi]

puzzling to modern sensibilities for two reasons

♦ based on a superstition that we do not believe to be true – we do not believe that the guilty are any more likely to drown than the innocent

♦ if people can be easily persuaded to hold a superstitious belief, why such an elaborate mechanism? Why not simply assert that those who are guilty will be struck dead by lightning?

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we attack these puzzles from the perspective of the theory of learning in games; use a model from our 1993 "Steady State Learning" paper

♦ partial characterization of patiently stable outcomes that arise as the limit of steady states with rational learning as players become more patient

♦ leads to a refinement of Nash equilibrium which also have self-confirming beliefs at certain information sets reachable by a single deviation

♦ which superstitions survive this refinement?

♦ according to this theory Hammurabi had it exactly right: his law uses the greatest amount of superstition consistent with patient rational learning

Overview of the Model

♦ society consists of overlapping generations of finitely lived players

♦ indoctrinated into the social norm as children "if you commit a crime you will be struck by lightning"

♦ enter the world as young adults with prior beliefs that the social norm is true

♦ being young and relatively patient, having some residual doubt about the truth of what they were taught, and being rational Bayesians, young players optimally decide to commit a few crimes to see what will happen

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lightning-strike norm

♦ most young players discover that the chances of being struck by lightning are independent of whether they commit crimes, and so go on to a life of crime, thereby undermining the norm

Hammurabi case

♦ the social norm is to not commit crimes; to only accuse the guilty; and to jump in the river when accused of a crime

♦ young players commit crimes, are accused of crimes, jump in the river and are punished; they learn that crime does not pay, and as they grow older stop committing crimes

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what about young accusers?

♦ will they experiment with false accusations, and learn that the river is as likely to punish the innocent as the guilty?

♦ accusers only get to play the game after a crime takes place

♦ there are few crimes, hence accusers only get to play infrequently

♦ infrequent play reduces the option value of experimentation because there will likely be a long delay before the knowledge gained can be put to use – hence no experimentation once off the equilibrium path

The Hammurabi Games

Example 2.1: The Hammurabi Game
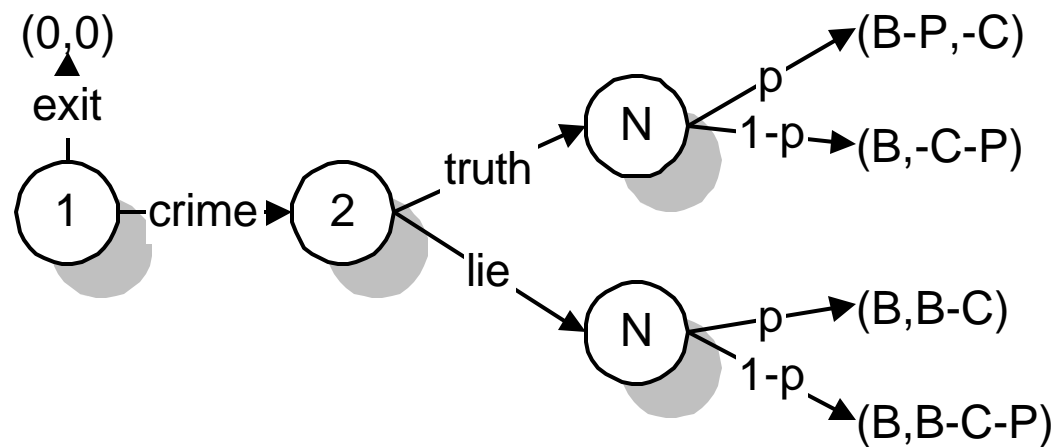
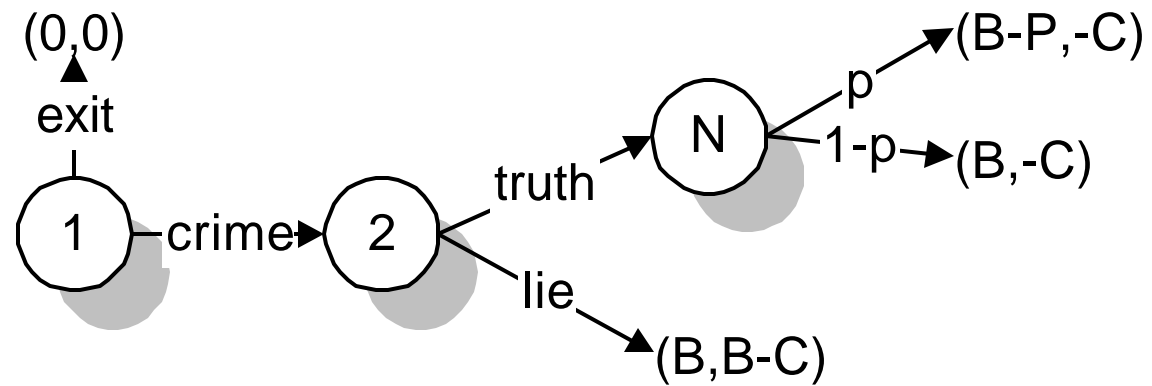Player 1 is a suspect; player 2 an accuser

C social cost of the **crime**

benefit to accuser of a false accusation, or **lie**, B the same as the benefit of the **crime** to the suspect
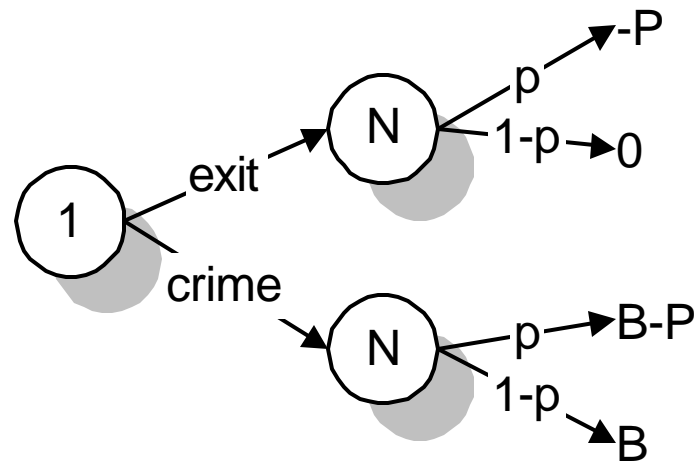
the cost of punishment P same for both

assume that $B < pP$ so probability of punishment sufficient to deter **crime**

Example 2.2: The Hammurabi Game Without a River

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Example 2.3: The Lightning Game

configurations in which there is no crime

Hammurabi game (Nash, but wrong beliefs about off-off path play)

♦ accuser tells the **truth** because he believes that if he **lie**s he will be punished with probability 1

Hammurabi game without a river (Nash, but not off-path rational)

♦ accuser tells the **truth,** and is indifferent (ex ante, not ex post)

lightning game (self-confirming, but not Nash)

♦ everyone believes that if they commit a **crime** they will be punished with probability 1, and that if they **exit** they will be punished with probability p

Simple Games

a *simple game*

♦ perfect information (each information set is a singleton node)

♦ each player has at most one information set on each path through the tree. (may have more than one information set, but once he has moved, he never gets to move again)

no own ties

will use a generic "no-tie" condition

no ties at all rules out the Hammurabi game with a river: the suspect
only cares whether he is punished or not, and there are a number of
ways he may fail to be punished

no player has two different actions at an information set that can
possibly result in a tie in his own payoff

in Hammurabi: the ties are for the suspect, but all occur when he
chooses to commit a crime, so two distinct own actions are not involved

no own ties implies that a player playing in the final stage of the game
has a unique best choice, and by backwards induction, every perfect
information game with no own ties has a unique subgame perfect
equilibrium

The Model

nodes in game tree $x \in X$, terminal nodes $z \in Z \subset X$

feasible actions at information sets $A(x)$

pure strategies $s_i \in S_i$, mixed $\boldsymbol{s}_i$, the state is $\boldsymbol{q}$ a mixed profile interpreted as fraction of population playing different pure strategies

payoffs $u_i : Z \to \Re$

I players plus Nature $(I + 1)$

Nature plays a fixed and given mixed strategy $\boldsymbol{s}^0_{I+1}$

reachable nodes $Z(s_i), X(s_i),\ X(\boldsymbol{s}_i)$

nodes reached $\bar{X}(\boldsymbol{s})$ (the "equilibrium path")

behavior strategies $\boldsymbol{p}_i$

beliefs about his opponents' play

m_i a probability measure over Π_{-i}, the set of other players' behavior strategies

beliefs are *independent*: players do not believe that there is a correlation between how an opponent plays at different information sets, or how different opponents play

$p_i(x \mid m_i)$ marginal induced by beliefs

preferences:

$$u_i(s_i, m_i) \equiv u_i(s_i, p_i(\cdot \mid m_i)) \equiv \sum_{z \in Z(s_i)} p_i(z \mid m_i) u_i(z).$$

when m_i is has a continuous density g_i we write $p_i(x \mid g_i), u_i(s_i, g_i)$.

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Subgame Confirmed Nash Equilibrium

Definition 4.1 : $\bar{q}$ *is a* self-confirming equilibrium *if for each player i and for each* s_i *with* $\bar{q}_i(s_i) > 0$ *there are beliefs* $m_i(s_i)$ *such that at every* $x \in \bar{X}(s_i, p_{-i})$,

(a) s_i *is a best response at* x *to* $m_i(s_i)$ *and*

(b) $m_i(s_i)$ *is correct.*

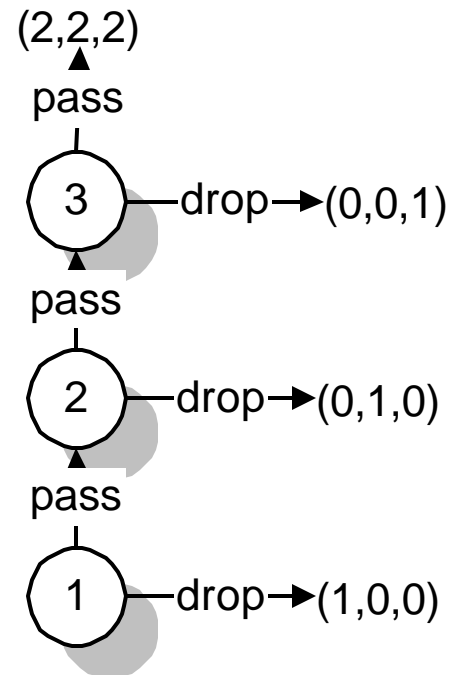Note also that *Nash equilibrium* strengthens (b) to hold at all information sets.

Definition 4.2: *In a simple game, node x is* one step off the path *of p if it is an immediate successor of a node that is reached with positive probability under p. Profile p is a* subgame-confirmed Nash equilibrium *if it is a Nash equilibrium and if, in each subgame beginning one step off the path, the restriction of p to the subgame is self-confirming in that subgame.*

In a simple game with no more than two consecutive moves, self-confirming equilibrium for any player moving second implies optimal play by that player, so subgame-confirmed Nash equilibrium implies subgame perfection.

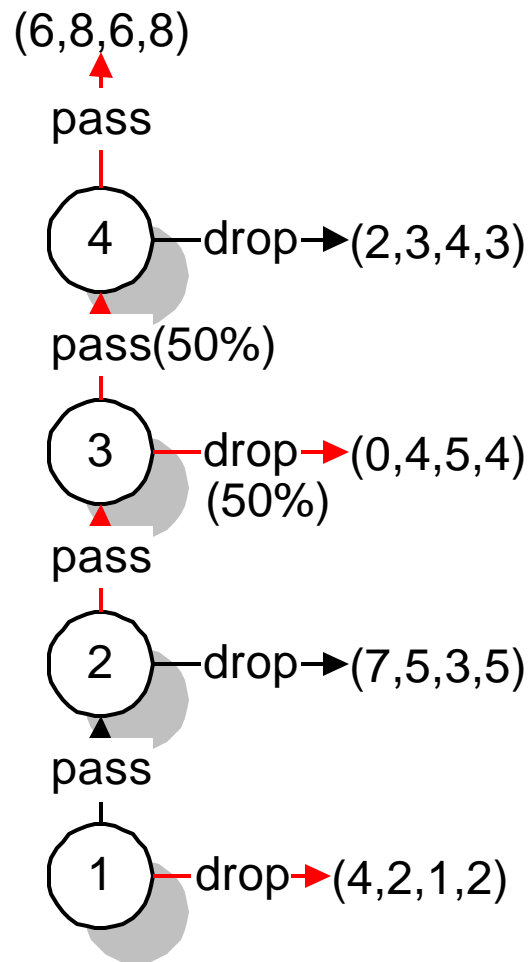can fail when there are three consecutive moves.

Example 4.1: The Three Player Centipede Game

```
(2,2,2)
   ▲
  pass
   │
  ┌─┐
  │3│──drop──►(0,0,1)
  └─┘
   ▲
  pass
   │
  ┌─┐
  │2│──drop──►(0,1,0)
  └─┘
   ▲
  pass
   │
  ┌─┐
  │1│──drop──►(1,0,0)
  └─┘
```

unique subgame-perfect equilibrium: all players to **pass**

(**pass**, **drop**, **pass**) is subgame-confirmed

subgame-confirmed Nash equilibrium is not equivalent to the requirement that the profile yield a *Nash* equilibrium at every node that is one step off of the path.

Example 4.2 The Four-player Centipede Game

(6,8,6,8)

pass

(4)—drop→(2,3,4,3)

pass(50%)

(3)—drop→(0,4,5,4)
(50%)

pass

(2)—drop→(7,5,3,5)

pass

(1)—drop→(4,2,1,2)

red is subgame confirmed

subgame-confirmed Nash equilibrium in which player 1 **drop**s out, player 3 must randomize

"red" equilibrium with player 3 randomizing 50-50 not path equivalent to to equilibrium with Nash play at all nodes at most one step off of the path of play

self-confirming equilibria of the subgame starting with player 2's move that are consistent with player 1 **drop**ping require player 3 to randomize.

conflict between player 1's and player 2's incentive constraints

for both to play as specified, player 3 must randomize

in a Nash equilibrium of subgame starting with 2's move, if player 2 passes and player 3 randomizes, player 4 must **pass,** so 3 must **pass** with probability 1

Rational Steady-State Learning

The Agent's Decision Problem

"agent" in the role of player *i* expects to play game T times wishes to maximize

$$\frac{1 - d}{1 - d^T} E \sum_{t=1}^{T} d^{t-1} u_t$$

u_t realized stage game payoff

agent believes that he faces a fixed time invariant probability distribution of opponents' strategies, unsure what the true distribution is

Definition 5.1: *Beliefs m_i are* non-doctrinaire *if m_i is given by a continuous density function g_i strictly positive at interior points.*

Note that allow priors can go to zero on the boundary, as is the case for many Dirichlet priors

assume non-doctrinaire prior g_i^0

$g_i(\cdot \mid z)$ posterior starting with prior g_i after z is observed

agents are assumed to play optimally

(dynamic programming problem defined in the paper)

histories are Y_i

optimal policy a map $r_i : Y_i \rightarrow S_i$ (may be several)

Steady States in an Overlapping generations model

♦ a continuum population

♦ doubly infinite sequence of periods

♦ generations overlap

♦ $1/T$ players in each generation

♦ $1/T$ enter to replace the $1/T$ player who leave

♦ each agent is randomly and independently matched with one agent from each of the other populations

each population assumed to use a common optimal rule r_i

given population fractions of each population playing pure strategies $\bar{q}_i(s_i)$

Using r we work out the fraction of the population with each experience $q_i(y)$

then recompute the fractions playing different strategies

$$f_i[\bar{q}](s_i) = \sum_{\{y_i|r_i(y_i)=s_i\}} q_i(y_i)$$

This is a polynomial map from the space of mixed strategy profiles to itself

a fixed point exists, and these fixed points are *steady states*.

a sequence of steady states $\lim_{T\to\infty} \overline{q}^T \to \overline{q}$ we say that $\overline{q}$ is a g^0, d-*stable state*

If $\overline{q}(d)$ are g^0, d-*stable states* and $\lim_{d\to 1} \overline{q}(d) \to \overline{q}$, we say that $\overline{q}$ is a *patiently stable state.*

Theorem 5.1: *(Fudenberg and Levine [1993b])* g^0, d-steady states are self-confirming; patiently stable states are Nash.

Patient Stability in Simple Games

two profiles $\bar{q}, \bar{q}'$ are *path equivalent* if they induce the same distribution over terminal nodes.

Theorem 6.1: *In a simple game, a patiently stable state $\bar{q}$ is path equivalent to a subgame-confirmed Nash equilibrium.*

corollary of a more general theorem proven in [2002b]

key result of this paper is a converse for simple games

important for the Hammurabi games

♦ shows that without the river, the crime-free equilibrium is not patiently stable

♦ however this can be shown by more elementary methods

♦ telling the truth is weakly dominated and generates no useful information; with non-doctrinaire beliefs it is not played in any steady state

♦ hence not played in the limit; but it is not a Nash equilibrium for player 1 to exit when player 2 is not telling the truth

a profile is *nearly pure* if Nature does not randomize on the equilibrium path, and no player except Nature randomizes off the equilibrium path

our proposed Hammurabi game profile is nearly pure – only Nature randomizes, and only off the equilibrium path

Theorem 6.2: *In simple games with no own ties, a subgame-confirmed Nash equilibrium that is nearly pure is path equivalent to a patiently stable state.*

This answers the Hammurabi puzzle: the Hammurabi equilibrium with the river is patiently stable; without the river it is not, nor is the lightning equilibrium stable

Games with Length at Most Three

a game has "length at most three" if no path through the tree hits more than three information sets

Theorem 6.3 *In simple games with no own ties, no Nature's move and length at most three, a subgame-confirmed Nash equilibrium is path equivalent to a patiently stable state.*

because in these games all equilibria are nearly pure

Lemma 6.1: In simple games with no own ties, no Nature's move and length at most three, a subgame-confirmed Nash equilibrium is path equivalent to a subgame-confirmed Nash equilibrium in which players play pure strategies.

in turn follows from

Lemma 6.2: *In simple games with no own ties, no Nature's move and length at most two, every self confirming equilibrium is path equivalent to a public randomization over Nash equilibria.*