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THE ECONOMICS OF INDETERMINACY IN OVERLAPPING GENERATIONS MODELS

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Recent research indicates that there are robust examples of overlapping generations economies in which there are indeterminate equilibria without fiat money and equilibria with more than one dimension of indeterminacy. This paper presents simple examples of a stationary, pure exchange overlapping generations economy with one good in each period and a representative consumer, who lives for three periods, in each generation. These examples exhibit every possible form of indeterminacy and instability. Furthermore, the parameters of the principal example agree with empirical evidence. We use our examples as case studies for analyzing the problems involved in computing the equilibria of such economies.

1. Introduction

This paper explores the implications of indeterminacy of equilibrium in overlapping generations models for applied work, for example, the study of the dynamics of fiscal policy. In particular, we attempt to answer three related questions: Do examples of indeterminacy depend on implausible parameter values? How can indeterminacy be diagnosed? How does indeterminacy manifest itself in truncated versions of infinite-horizon models?

That an overlapping generations economy might have a continuum of equilibria is well known. When counting the equations and unknowns in his equilibrium conditions, Samuelson (1958, p. 470) notes that 'we never seem to get enough equations: lengthening our time period turns out always to add as many new unknowns as it supplies equations'. Unfortunately, most discussions of indeterminacy of equilibrium have focused on a special model, one with a single good in each period and consumers who live for only two

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periods. This model has special features: indeterminacy occurs only if there is a nonzero amount of nominal debt, fiat money. Furthermore, indeterminacy is one-dimensional in the sense that equilibria can be locally indexed by the real value of fiat money.

To illustrate these points, let us consider a model in which there are two generations alive in the first period, a generation that lives only in period 1 and holds an amount of fiat money, m, which may be positive, negative, or zero, and a generation that lives in periods 1 and 2. The equilibrium condition can be written

$$f_1(p_1, p_2, m) = 0. \tag{1.1}$$

Here f_1 is the aggregate excess demand function of the two generations and p_1 and p_2 are prices. In period t, $t = 2, 3, \ldots$, there are two generations, one that lives in periods $t-1$ and t and one that lives in periods t and $t+1$. The equilibrium condition can be written

$$f(p_{t-1}, p_t, p_{t+1}) = 0. \tag{1.2}$$

The excess demand functions in (1.1) and (1.2) are homogeneous of degree zero in their arguments.

To see the possibility of indeterminacy, let us count equations and unknowns. If we normalize prices by fixing p_1, then (1.1) consists of one equation and two unknowns, p_2 and m. After that, each condition adds one equation and one unknown. There is, therefore, one degree of freedom: if we fix m/p_1, the real value of fiat money, then we use (1.1) to solve for p_2. Knowing p_1 and p_2, we can use (1.2) to solve for p_3, and so on. There are two potential problems with this procedure for constructing equilibria: for some values of p_{t-1} and p_t, there may be no positive value of p_{t+1} that solves (1.2), and for other values of p_{t-1} and p_t, there may be more than one value of p_{t+1} that solves (1.2). As we shall see, however, both of these problems can be resolved by restricting attention to equilibrium price paths that start and remain in a neighborhood of the stationary solution to (1.2).

This model has the special feature that fixing the initial real level of fiat money reduces the system to one with no degrees of freedom. This is the intuition behind the results of Gale (1973), who shows that in an economy with a single, two-period-lived consumer in each generation and a single good in each period, indeterminacy is one-dimensional and depends on the existence of fiat money. Balasko and Shell (1981) extend these results to an economy in which there are many goods in each period but a single, two-period-lived consumer with a Cobb–Douglas utility function in each generation [see also Samuelson (1960)]. Geanakoplos and Polemarchakis (1984) and Kehoe and Levine (1984a) have extended these results to an economy with a single, two-period-lived consumer with intertemporally separable preferences in each generation. Kehoe, Levine, Mas-Colell and Woodford (1986) have extended them to a general economy with gross substitutes demands. Calvo (1978) has also constructed examples in which there is a one-dimensional indeterminacy indexed by the value of an asset like land or capital.

In contrast, Kehoe and Levine (1984b, 1985) have shown that such strong results do not obtain without strong assumptions. They study a stationary pure exchange economy with n goods in each period and general demand functions. They prove that with a nonzero stock of nominal debt there is, potentially, an n-dimensional indeterminacy, while with no nominal debt there is potentially an $n-1$-dimensional indeterminacy. In particular, indeterminacy does not depend on the existence of fiat money or other assets.

The intuition behind the results in the more general model can be seen by reinterpreting equilibrium conditions (1.1) and (1.2). In a model with n goods the price p_t and excess demands $f_1(p_1, p_2, m)$ and $f(p_{t-1}, p_t, p_{t+1})$ are now n-dimensional vectors. Counting equations and unknowns, we see that (1.1) is a system of n equations in $2n+1$ unknowns. For each t, (1.2) adds n unknowns and n equations. Once we have imposed a price normalization, we are left with a system with n degrees of freedom. If we set $m=0$, there are only $n-1$ degrees of freedom left. The argument in Kehoe and Levine (1984b) depends on a result due to Debreu (1974) that says that the excess demand function of a generation is arbitrary except for continuity, homogeneity, Walras's law, and a boundary condition: for any excess demand function that satisfies these properties there exists a generation of $2n$ utility-maximizing consumers who generate it. This leaves open the question of how far we have to go to construct robust examples in which there are indeterminate equilibria without fiat money or equilibria with more than one dimension of indeterminacy. Do such examples depend on implausible parameter values?

In this paper we present examples in which the only departure from the economy considered by Gale (1973) is that the single consumer in each generation lives three, rather than two, periods. In these examples the elasticity of intertemporal substitution in consumption over time is 0.25. This accords well with the empirical evidence presented in Mankiw, Rotemberg and Summers (1985) and Auerbach and Kotlikoff (1987), for example. The representative consumer discounts future consumption by about 3.5 percent per year. In addition, individual endowments exhibit a substantial hump in the middle period. In this economy there is both a one-dimensional family of efficient equilibria without valued fiat money that converge to an efficient steady state and a two-dimensional family of efficient equilibria with valued fiat money that also converge to an efficient steady state. With a higher endowment hump in the middle period, we find a two-dimensional family of inefficient equilibria, all converging to an inefficient steady state in which money has no value. (In all of these cases Pareto efficiency is associated with interest rates that are asymptotically negative.)

An important property of an intertemporal elasticity of substitution of 0.25 is that it allows goods in different periods to be gross complements at some prices. In situations where all goods are gross substitutes, Kehoe, Levine, Mas-Colell and Woodford (1986) have shown that indeterminacy of the type discussed in this paper is impossible. To guarantee gross substitutability, however, the elasticity of substitution must be greater than or equal to 1.0, which is an implausibly high value.

Indeterminacy poses problems for the researcher interested in doing comparative statics analysis: the specification of the environment does not suffice to determine the equilibrium even locally near a historically given equilibrium. The examples in this paper serve as case studies of how to compute the equilibria of an intertemporal general equilibrium model and how to diagnose indeterminacy. An economist working with this type of model would, in theory, need to solve a system with an infinite number of equations and with an infinite number of unknowns. Except for very special cases this is an impossible task.

In practice, such economists face three alternatives:

First, they could solve for a steady state, a solution to the equilibrium conditions that ignores any initial conditions and remains constant, or grows proportionally, over time. [This is the approach taken by Diamond (1965) and Feldstein (1977), for example.] In a pure exchange economy with two-period-lived consumers and n goods, for example, a steady state is a price vector p and an inflation factor β such that

$$f(p, \beta p, \beta^2 p) = 0. \tag{1.3}$$

One advantage of this alternative is, as Kehoe and Levine (1984b) demonstrate, that steady states of such models are, in general, determinate. They are also relatively easy to compute. The disadvantage is that it ignores initial conditions and the transition from one steady state to another.

Second, the economists could linearize the equilibrium conditions around a steady state (or cycle) and then solve the linearized version of the model. Thus, they could exactly solve an approximate version of the model. [This is the approach taken by Laitner (1984), for example.] Because of homogeneity, we can linearize the equilibrium condition (1.2) as

$$D_1 f p_{t-1} + D_2 f p_t + D_3 f p_{t+1} = 0. \tag{1.4}$$

Here each $n \times n$ matrix of partial derivatives $D_i f$ is evaluated at the steady state $(p, \beta p, \beta^2 p)$. Rewriting this linearized equilibrium condition as a first-order difference equation, we obtain:

$$\begin{bmatrix} p_t \\ p_{t+1} \end{bmatrix} = \begin{bmatrix} 0 & I \\ -D_3 f^{-1} D_1 f & -D_3 f^{-1} D_2 f \end{bmatrix} \begin{bmatrix} p_{t-1} \\ p_t \end{bmatrix}. \tag{1.5}$$

Indeterminacy of the linearized system manifests itself as too many stable eigenvalues of the $2n \times 2n$ matrix in (1.5). The advantage of this alternative is that indeterminacy of the linearized system is easy to diagnose and equilibria are easy to compute: both are simple matters of linear algebra. Furthermore, the local stable manifold theorem says that the qualitative properties of the nonlinear system near the steady state are almost always the same as those of the linearized system. In particular, indeterminacy in one system almost always corresponds to indeterminacy in the other. (Violations of the non-degeneracy conditions implicitly assumed in the 'almost always' caution can, as we shall see, be ignored in practice; in any case such violations would be obvious in the linearization itself.) The disadvantage of this alternative is that equilibria of the linearized system may be very poor approximations to equilibria of the nonlinear system far from the steady state.

Third, the economists could truncate the model after a long but finite horizon. They would then be faced with a system with a large but finite number of equations. To solve this system, they would need to impose terminal conditions on some of the prices, for example, by fixing them at their steady-state values. Thus, the economists could find an approximate solution to the nonlinear model. [This is the approach taken by Auerbach and Kotlikoff (1987), for example.] Truncating the model at period T, we obtain, for example,

$$f(p_{T-1}, p_T, \|p_T\| \beta p) = 0. \tag{1.6}$$

The advantage of this alternative is that the equilibria that it computes are good approximations to the equilibria of the infinite-horizon model, at least if the truncation date T is large enough. One disadvantage of this alternative is that it is relatively more difficult to compute approximate equilibria using this approach than it is using the other two. A more serious disadvantage is that it is much more difficult to diagnose indeterminacy, or even lack of convergence to the steady state, in the infinite-horizon, nonlinear model using this approach than it is using the linearized system.

These three approaches to computing equilibria are, perhaps, best viewed not as alternatives, but as complementary research tools. To explore their relative advantages and disadvantages, we present a series of simple and highly stylized examples. The central example has a very special feature that makes it ideal for comparing alternatives for computing equilibria: because it exhibits the maximum level of indeterminacy, exact equilibria of the infinite-horizon, nonlinear model can be computed. Initial prices can be chosen subject only to the constraint that they satisfy the equilibrium conditions in the initial two periods. Subsequent prices can be computed simply by working the conditions corresponding to (1.2) forward. Since all price paths close enough to the steady state converge to the steady state, we do not need to worry about terminal conditions.

Results on indeterminacy in models with infinite time horizons are primarily interesting insofar as they provide insights into economies with long but finite time horizons. Our examples illustrate how the indeterminacy of equilibrium in the infinite model corresponds to acute sensitivity of equilibrium to terminal conditions in the corresponding finite economy. Indeed, seemingly trivial variations in the characteristics of people who would not be born for 10 centuries can cause current annual interest rates to jump from -7 percent to nearly 5 percent. This should serve as a warning to applied economists who use these types of models to study public finance issues.

Our examples, although simple and stylized, are intended to illustrate alternative possibilities for indeterminacy in more complex and detailed models. These possibilities certainly do not disappear as models become more complex: Muller and Woodford (1988), for example, consider economies with production, infinitely lived assets, and mixtures of finitely lived and infinitely lived consumers. They find that, although the introduction of infinitely lived consumers or assets may rule out inefficient equilibria and equilibria with fiat money, it does not rule out the possibility of indeterminacy, except in extreme cases. Woodford (1986a) has further argued that indeterminacy can occur in models with a finite number of infinitely lived consumers who face borrowing constraints. Even the positive implications of gross substitutability in pure exchange economies disappear when production is introduced: Calvo (1978) presents a simple example of an overlapping generations economy with production in which the representative consumer in each generation can have gross substitutes excess demand but in which there can be a continuum of equilibria even though there is no fiat money. Furthermore, Spear (1988) provides an example of an economy with a single infinitely lived consumer in which there is a robust continuum of equilibria because of an externality. Kehoe, Levine and Romer (1989) provide a similar example in which indeterminacy is due to distortionary taxes.

As well as presenting practical problems to researchers interested in performing comparative statics experiments, the possibility of indeterminacy poses important conceptual problems. Most importantly, it undermines the concept of perfect foresight equilibrium. The agents in the model, like the modeler, cannot use the model itself to make determinate predictions about the future. Closely related to this is the possibility of sunspot equilibria, i.e.

equilibria that depend on otherwise extraneous random processes: if the model itself does not pin down agents' expectations about the future, room is left open for random coordinating devices. See Woodford (1986b), Laitner (1988), and Peck (1988) for discussions of the relationship between indeterminacy and sunspot equilibria.

2. The model

Consider a stationary economy in which the single consumer born in period t, $t=1,2,\dots$, lives for three periods and has the utility function

$$u(c_1,c_2,c_3)=\sum_{i=1}^{3} a^{i-1}(c_i^b-1)/b, \tag{2.1}$$

where a is the discount factor, b satisfies $b<1$, and c_i is the consumption in period $t+i-1$. This is, of course, the constant elasticity of substitution utility function with elasticity of substitution $\eta=1/(1-b)$. The consumer faces the budget constraint

$$\sum_{i=1}^{3} p_{t+i-1}c_i=\sum_{i=1}^{3} p_{t+i-1}w_i, \tag{2.2}$$

where (w_1,w_2,w_3) is the endowment stream. The corresponding excess demand functions are denoted $x_j(p_t,p_{t+1},p_{t+2})$, $j=1,2,3$. These functions are continuously differentiable for all strictly positive prices, are homogeneous of degree zero, and obey Walras's law:

$$\sum_{i=1}^{3} p_{t+i-1}x_i(p_t,p_{t+1},p_{t+2})=0. \tag{2.3}$$

In addition to these consumers, there are two others, an old consumer who lives only in period 2 and a middle-aged consumer who lives in periods 1 and 2. The old consumer, consumer -1, derives utility only from consumption of the single good in the first period, so we need not specify a utility function. The consumer has m_{-1} units of fiat money, which may be positive, negative, or zero and an excess demand function of

$$x_3^{-1}(p_1,m_{-1})=m_{-1}/p_1. \tag{2.4}$$

The middle-aged consumer, consumer 0, has the utility function

$$u_0(c_2, c_3) = \sum_{i=2}^{3} a^{i-2}(c_i^b - 1)/b, \tag{2.5}$$

an endowment stream (w_2^0, w_3^0) of goods, an endowment m_0 of fiat money, and excess demand functions of $x_j^0(p_1, p_2, m_0)$, $j = 1, 2$.

The equilibrium conditions for this economy are

$$x_3^{-1}(p_1, m_{-1}) + x_2^0(p_1, p_2, m_0) + x_1(p_1, p_2, p_3) = 0, \tag{2.6}$$

$$x_3^0(p_1, p_2, m_0) + x_2(p_1, p_2, p_3) + x_1(p_2, p_3, p_4) = 0, \tag{2.7}$$

$$x_3(p_{t-2}, p_{t-1}, p_t) + x_2(p_{t-1}, p_t, p_{t+1}) + x_1(p_t, p_{t+1}, p_{t+2}) = 0, \quad t = 3, 4, \ldots. \tag{2.8}$$

Let $m = m_{-1} + m_0$. A straightforward calculation using the equilibrium conditions and Walras's law shows that

$$m = -p_t x_1(p_t, p_{t+1}, p_{t+2}) - p_{t+1} x_2(p_t, p_{t+1}, p_{t+2}) - p_{t+1} x_1(p_{t+1}, p_{t+2}, p_{t+3}), \tag{2.9}$$

for all t; just as in the two-period-lived model, the amount of fiat money stays constant over time.

To see the possibility of indeterminacy in this economy, let us count equations and unknowns. If we fix m_{-1} and m_0, then the first two equilibrium conditions consist of two equations in the four unknowns p_1, p_2, p_3, and p_4. After that, each condition adds one equation and one unknown. There are, therefore, two degrees of freedom. In the case where $m_{-1} = m_0 = 0$, however, the equilibrium conditions are homogeneous in prices, and so we can impose a price normalization to reduce this to one degree of freedom. In the case where $m_{-1} = -m_0 \neq 0$ there are still two degrees of freedom. We shall ignore this case, however, and mention it only to warn the reader that distributional effects as well as fiat money can be responsible for extra dimensions of indeterminacy.

One problem with simply counting equations and unknowns is that we do not know whether we can construct a price path for all values of p_1, p_2, p_3, and p_4 that satisfy the equilibrium conditions in the first two periods. To avoid this problem, Kehoe and Levine (1984b, 1985) focus attention on price paths that converge to a steady state. Such paths are the easiest to study. Price paths that do not converge to a steady state may display very complex periodic or even chaotic behavior [see, for example, Benhabib and Day

(1982)]. It may be difficult to distinguish them from price sequences that satisfy the equilibrium conditions for a long time, but eventually lead to prices that are negative, where excess demands explode, or to prices where, for some other reason, continuation of the sequence is impossible. We could argue, however, that the computational difficulties associated with paths that do not converge to steady states make them implausible as perfect foresight equilibria.

Another problem with counting equations and unknowns is that, given values of p_{t-2}, p_{t-1}, p_t, and p_{t+1}, there may be more than one value of p_{t+2} that satisfies (2.8). To avoid this problem, Kehoe and Levine (1984b, 1985) consider only price paths that remain in some open neighborhood of a steady state where the derivative of x_1 with respect to its third argument, $D_3x_1(1,\beta,\beta^2)$ is nonzero. Kehoe and Levine (1984b) prove that $D_3x_1(1,\beta,\beta^2)$ is, in fact, nonzero at almost every steady state β.

A steady state of this economy is an inflation factor $\beta>0$ such that $p_t=\beta^t$ satisfies the equilibrium conditions from the second period onwards. Here $r=1/\beta-1$ is the steady-state interest rate. Just as in the case with two-period-lived consumers, there are two types of steady states, real steady states in which $\beta\neq 1$ and $m=0$ and nominal steady states in which $\beta=1$ and $m\neq 0$. Kehoe and Levine (1984b) prove that $\beta=1$ and $m=0$ occur at the same steady state only for a closed, nowhere dense set of economies in the appropriate topology. They also prove that generically there exist an odd number of steady states of each type. (With only one good in each period this result has no bite for nominal steady states since $\beta=1$ is the unique nominal steady state.)

3. The linearized model

To study the behavior of equilibrium price paths near a steady state, we can linearize the equilibrium conditions:

$$(D_1x_3^{-1}+D_1x_2^0+D_1x_1)p_1+(D_2x_2^0+D_2x_1)p_2+D_3x_1p_3$$
$$=D_1x_3^{-1}+D_1x_2^0+D_2x_2^0\beta-x_3^{-1}-x_2^0-x_1, \tag{3.1}$$

$$(D_1x_3^0+D_1x_2)p_1+(D_2x_3^0+D_2x_2+\beta^{-1}D_1x_1)p_2+(D_3x_2+\beta^{-1}D_2x_1)p_3$$
$$+\beta^{-1}D_3x_1p_4=D_1x_3^0+D_2x_3^0\beta-x_3^0-x_2-x_1, \tag{3.2}$$

$$D_1x_3p_{t-2}+(D_2x_3+\beta^{-1}D_1x_2)p_{t-1}+(D_3x_3+\beta^{-1}D_2x_2+\beta^{-2}D_1x_1)p_t$$
$$+(\beta^{-1}D_3x_2+\beta^{-2}D_2x_1)p_{t+1}+\beta^{-2}D_3x_1p_{t+2}=0,\quad t=3,4,\ldots \tag{3.3}$$

Here all functions and derivatives are evaluated at $(1,\beta,\beta^2)$ and factors like β^{-1} show up because of homogeneity: since x_1 is homogeneous of degree zero, for example, D_1x_1 is homogeneous of degree minus one and $D_1x_1(\beta,\beta^2\beta^3)=\beta^{-1}D_1x_1(1,\beta,\beta^2)$.

The linearized versions of the two initial conditions (3.1) and (3.2) generically determine a two-dimensional affine set in $\mathbb{R}^4$. We want to determine the dimension of the intersection of this set with the subspace of vectors $(p_1,p_2,p_3,p_4)\in\mathbb{R}^4$ that lead to convergence to the steady state β when used as starting values for the difference eq. (3.3):

$$\lim_{t\to\infty}(p_t,p_{t+1},p_{t+2},p_{t+3})/p_t=(1,\beta,\beta^2,\beta^3). \tag{3.4}$$

The local stable manifold theorem of dynamical systems theory, as presented, for example, by Irwin (1980), says that, in general, what is true of the linearized system is true of the nonlinear system in some open neighborhood of the steady state. In particular, the intersection of the vectors that satisfy (3.1) and (3.2) with the stable subspace of (3.3) has the same dimension as the manifold of equilibria of the original system (2.6)–(2.8). In fact, this intersection is the best affine approximation to the equilibrium manifold at the steady state.

To determine the dimension of this stable subspace, we examine the roots of the polynomial

$$\begin{aligned}&D_1x_3+(D_2x_3+\beta^{-1}D_1x_2)\lambda+(D_3x_3+\beta^{-1}D_2x_2+\beta^{-2}D_1x_1)\lambda^2\\&\quad+(\beta^{-1}D_3x_2+\beta^{-2}D_2x_1)\lambda^3+\beta^{-2}D_3x_1\lambda^4=0.\end{aligned} \tag{3.5}$$

These roots can also be viewed as eigenvalues of the 4×4 matrix formed when we convert (3.3) into a first-order difference equation in four variables. Since x_1, x_2, and x_3 are homogeneous of degree zero, $\lambda=\beta$ is a root, and since they satisfy Walras's law, $\lambda=1$ is a root. We need to divide by β every period to make $(1,\beta,\beta^2,\beta^3)$ a fixed point of the linear system. Consequently, the stability condition is $|\lambda|<\beta$. Let n^s be the number of stable roots; n^s is equal to 0, 1, 2, or 3. The stable subspace has dimension n^s+1. The extra dimension shows up because of homogeneity and corresponds to the root β: if (p_1,p_2,p_3,p_4) leads to convergence to the steady state, then so does $(\theta p_1,\theta p_2,\theta p_3,\theta p_4)$. The intersection of this subspace with the set that satisfies (3.1) and (3.2) generically has dimension $(n^s+1)+2-4=n^s-1$, which can be negative, 0, 1, or 2. If it is negative, then in general no solution exists that converges to the steady state; that is, the steady state is unstable. If it is 0, then the solutions are locally unique; that is, equilibria are determinate.

In the case where $m_{-1}=m_0=0$, we can reduce the dimension by only considering the values of (p_1, p_2, p_3, p_4) for which $p_1 x_1(p_1, p_2, p_3) + p_2 x_2(p_1, p_2, p_3) + p_2 x_1(p_2, p_3, p_4) = 0$. Kehoe and Levine (1984b, 1985) argue that in this case the root $\lambda = 1$ is irrelevant and that the dimension of the relevant intersection is $\bar{n}^s - 1$, where $\bar{n}^s$ is this number of roots less than β in modulus excluding (possibly) $\lambda = 1$. Consequently, this dimension can be negative, 0, or 1. Again, if it is negative, then no solution exists in general.

The root $\lambda = 1$ is crucial at real steady states if $m_{-1} + m_0 \neq 0$, however. If $\beta > 1$, then this is a stable root, and paths with nominal debt can converge to the real steady state because inflation causes the real value of this debt to approach zero. In contrast, if $\beta < 1$, then this is an unstable root, and no path with nominal debt can converge because deflation causes the real value of this debt to approach infinity.

There are three possible problems involved in using the linearized version of the model.

First, our use of the phrase 'in general' indicates that we require three regularity conditions to be satisfied: (1) $D_3 x_1(1, \beta, \beta^2)$ cannot equal zero at the steady state β; (2) the characteristic polynomial (3.5) cannot have any root, except the one implied by homogeneity, with modulus β; and (3) the intersection of the affine set of initial conditions that satisfy the equilibrium conditions in the first two periods, (3.1) and (3.2), with the stable subspace of the linearized system, (3.3), must have the dimension suggested by counting equations and unknowns. If the regularity conditions are violated, then we can no longer claim that what is true of the linearized system is true of the nonlinear system. Kehoe and Levine (1984b) prove the regularity conditions are generic, that is, satisfied by almost all economies in a precise mathematical sense.

Second, the local manifold theorem says that what is true of the linearized system is true of the nonlinear system only in an open neighborhood of the steady state. It does not tell us how big this open neighborhood is in practice. On one hand, there may be no equilibrium price paths with (p_1, p_2, p_3, p_4) outside this neighborhood. On the other, a price path may leave this neighborhood of the steady state but may nevertheless generate an equilibrium price path.

Third, although the behavior of the linearized system may provide a good guide to the qualitative behavior of the nonlinear system, it may be significantly different quantitatively. In other words, the neighborhood of the steady state in which a solution to the linearized version of the model can be used as a good approximation to the equilibrium may be even smaller than the neighborhood in which the local stable manifold theorem applies.

In practice, the first of these possible problems can be ignored. No computer program, for example, would ever find more than one root of (3.5) with modulus exactly equal to β. (Some numerical difficulties might arise, however, in situations where another root has modulus very close to β.) In

Table 1

	Steady state		
Initial money	$\beta<1$	$\beta=1$	$\beta>1$
$m=0$	Unstable, 0, 1	Unstable	Unstable, 0, 1
$m\neq 0$	Unstable	Unstable, 0, 1, 2	0, 1, 2

Table 2

	Steady state		
Initial money	$\beta<1$	$\beta=1$	$\beta>1$
$m=0$	0	Unstable	0
$m\neq 0$	Unstable	0, 1	0, 1

any case, it is trivial to verify that the examples presented in the next section satisfy the three regularity conditions. In general, it is difficult to assess the importance of the second and third possible problems in practice. Since the first example presented in the next section has the special feature that we can calculate its equilibria exactly, however, we can calculate the size of open neighborhoods in which the nonlinear system behaves like the linearized system and compare solutions of the linearized system with exact solutions to the nonlinear system.

4. Examples

In this section we present simple examples to illustrate the different possibilities for indeterminacy of equilibria. Let us begin by summarizing the possibilities. We distinguish among equilibria according to whether the steady states they converge to satisfy $\beta<1$, $\beta=1$, or $\beta>1$. This distinction has a close connection with Pareto efficiency: if $\beta\leqq 1$, then the interest rate $r_t=p_t/p_{t+1}-1$ is asymptotically non-negative and the equilibrium is efficient. If $\beta>1$, however, then the interest rate is asymptotically negative and the equilibrium is inefficient. See Balasko and Shell (1980) and Burke (1987) for discussions of this familiar efficiency criterion in overlapping generations economies. Table 1 illustrates the various possibilities for dimensions of indeterminacy. In contrast, table 2 illustrates the possibilities for the model with two-period-lived consumers. In our examples we focus on the novel possibilities in the three-period-lived model, in particular, the possibility of indeterminacy even when $m=0$.

Consider first an economy in which $a=0.5$, $b=-3$, and $(w_1,w_2,w_3)=(3,12,1)$. To interpret these parameters, think of each period as being 20 years in length. Since $0.5=(0.96594)^{20}$, the representative consumer in this

Table 3

Steady states	β	Other roots		
1	0.17563	1	0.00130	0.57803
2	0.79247	1	$0.32315\pm$	$0.50807i$
3	1	0.79000	$0.40969\pm$	$0.69917i$
4	44.63396	1	1.56835	-144.10262

economy has a discount rate of roughly 3.5 percent per year and an intertemporal elasticity of substitution of $1/(1+3)=0.25$, which is the value chosen by Auerbach and Kotlikoff (1987). The consumer also has a humped life-cycle earnings profile. Although this example is fanciful, the parameters chosen are the sorts of numbers that would be calibrated from empirical evidence.

This economy has one nominal and three steady states. To determine the roots of the fourth-order polynomial (3.5) at each steady state, we start by evaluating the excess demand function (2.3) at $(p_t, p_{t+1}, p_{t+2})=(1,\beta,\beta^2)$. At the steady state where $\beta=1$,

$$\begin{bmatrix} D_1x_1 & D_2x_1 & D_3x_1 \\ D_1x_2 & D_2x_2 & D_3x_2 \\ D_1x_3 & D_2x_3 & D_3x_3 \end{bmatrix} = \begin{bmatrix} -2.24080 & 3.15531 & -0.91451 \\ -0.56420 & 1.33320 & -0.76900 \\ -0.47443 & 2.23114 & -1.75671 \end{bmatrix}. \tag{4.1}$$

Notice that, as we expected, $D_3x_1\neq 0$. [Notice too that, since this matrix has some negative off-diagonal elements, (x_1,x_2,x_3) violates gross substitutability.] The polynomial that we are interested in is

$$-0.47443+1.66694\lambda-2.66431\lambda^2+2.38631\lambda^3-0.91451\lambda^4=0. \tag{4.2}$$

One of the roots is, of course, $\lambda=1$. The other three are 0.79000, $0.40969+0.69917i$, and $0.40969-0.69917i$, as can easily be verified.

The roots at all four steady states are listed in table 3. The modulus of the pair of complex conjugates at the steady states where $\beta=0.79247$ is 0.60213; where $\beta=1$ it is 0.81036. Notice that, as we expected, no steady state has another root with modulus equal to β.

To construct an example of a continuum of Pareto-efficient equilibria without valued fiat money that converge to a common efficient steady state, we focus our attention on the steady state where $\beta=0.79247$. Let $m_{-1}=m_0=0$ and let $w_2^0=8.26762$ and $w_3^0=1$, so that the initial middle-aged generation has a smaller life-cycle hump than subsequent generations. It can be checked that $(p_1,p_2,p_3,p_4)=(1,0.79247,(0.79247)^2,(0.79247)^3)$ satisfies the conditions for equilibrium in the first two periods. Since $\beta=0.79247$ is a steady state, this is a legitimate equilibrium price path. Our earlier arguments imply that this is only one of a continuum.

This example has the very special feature that this steady state has the maximum possible dimension of indeterminacy. We can therefore choose any starting values for prices close enough to the steady state and use the equilibrium conditions to solve for an exact equilibrium of the nonlinear model. Since $m_{-1}=m_0=0$, the excess demands of generations -1 and 0 are homogeneous of degree zero. We can choose $p_2/p_1=0.79247+\varepsilon$ for any small ε, positive or negative, and use the equilibrium conditions (2.6) and (2.7) to solve for p_3/p_2 and p_4/p_3. Using the equilibrium condition (2.8), we can solve for an infinite price sequence. This price sequence must converge to one where $p_{t+1}/p_t=0.79247$ since the modulus of the root governing stability is less than 0.79247. The root $\lambda=1$ is, as we have explained, irrelevant since $m=0$ everywhere along this price path.

In fact, ε need not be very small: every p_2/p_1 in the interval $0.42075<p_2/p_1<14.16353$ determines a distinct equilibrium that converges to the steady state $\beta=0.79247$. In other words, the neighborhood of the steady state in which the local stable manifold theorem guarantees that the behavior of the linear system is a good guide to the qualitative behavior of the nonlinear system is very large in this example. Fig. 1 illustrates the range of possibilities; to keep the figure manageable a logarithmic scale is used. Notice that $p_2/p_1=0.42075$ determines an equilibrium that converges to the steady state where $\beta=0.17562$. Otherwise, all values of p_2/p_1 outside this interval determine paths that eventually lead to a negative price.

Fig. 2 illustrates three typical equilibria of our numerical example. From an economic perspective, these different equilibria exhibit a wide range of behavior. Since each period is about 20 years long, an intertemporal price ratio of p_{t+1}/p_t implies an annual interest rate of roughly $(p_{t+1}/p_t)^{-1/20}-1$. Equilibrium A has an annual interest rate of around -10.4 percent ($p_{t+1}/p_t=9.00$) for 20 years, which gradually increases to 7.4 percent ($p_{t+1}/p_t=0.24$) over the following 80 years. The interest rate then falls gradually back to the steady-state level of 1.2 percent. In contrast, equilibrium B has an interest rate of around 2.6 percent ($p_{t+1}/p_t=0.60$) for around 40 years. Over the next 40 years this falls gradually to 0.4 percent ($p_{t+1}/p_t=0.93$), then overshoots again to 1.5 percent ($p_{t+1}/p_t=0.75$) before settling near the steady state. The point to emphasize is that in the foreseeable future one equilibrium has an interest rate of -10.4 percent, while the other has an interest rate of 2.6 percent. This is true even though eventually these two equilibria converge to the same steady-state interest rate of 1.2 percent.

How good an approximation to the exact equilibria in fig. 2 is provided by their linear approximations? Fig. 3 shows each of these equilibria along with the path followed by its linear approximation. Notice that the approxi-

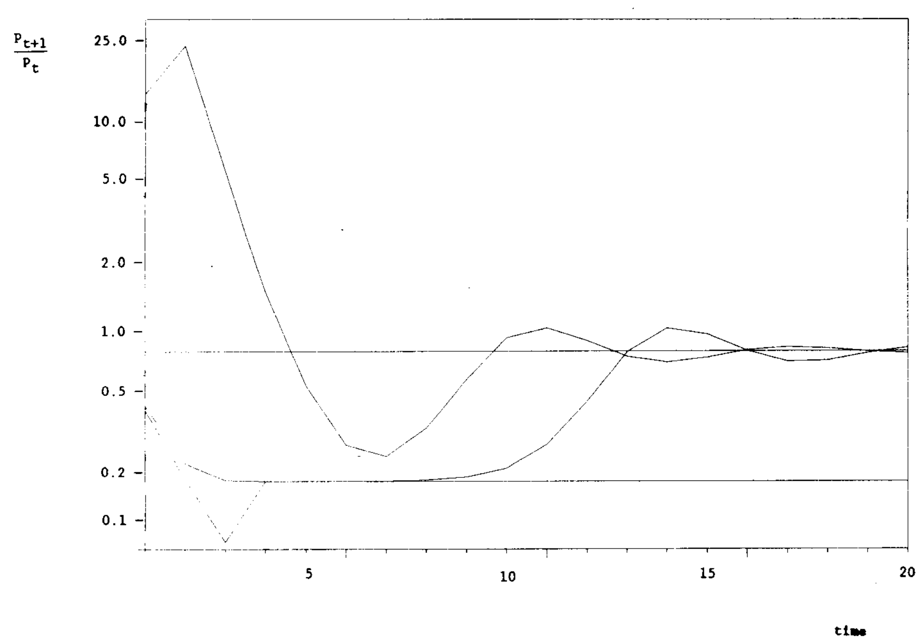

$\frac{p_{t+1}}{p_t}$
25.0
10.0
5.0
2.0
1.0
0.5
0.2
0.1
5
10
15
20
time

Fig. 1

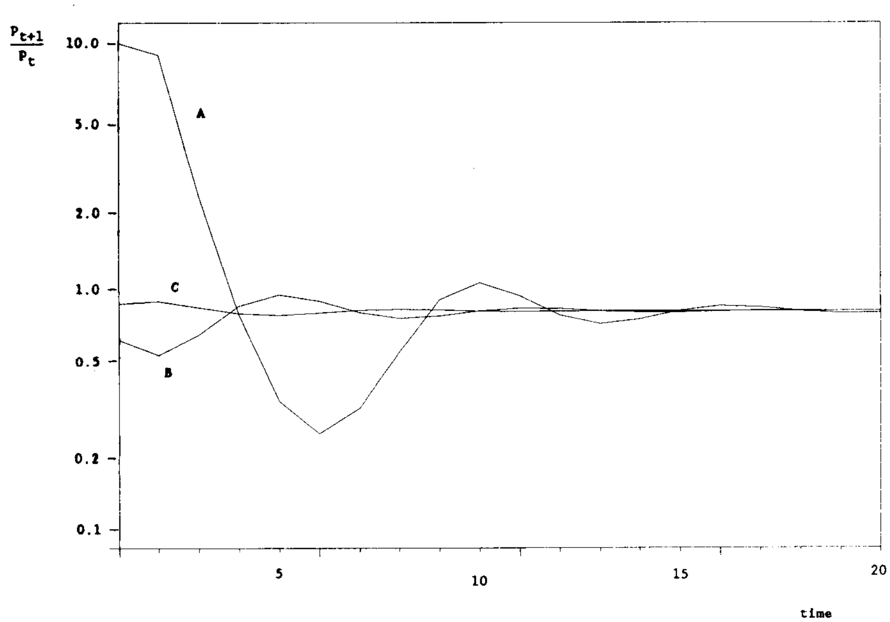
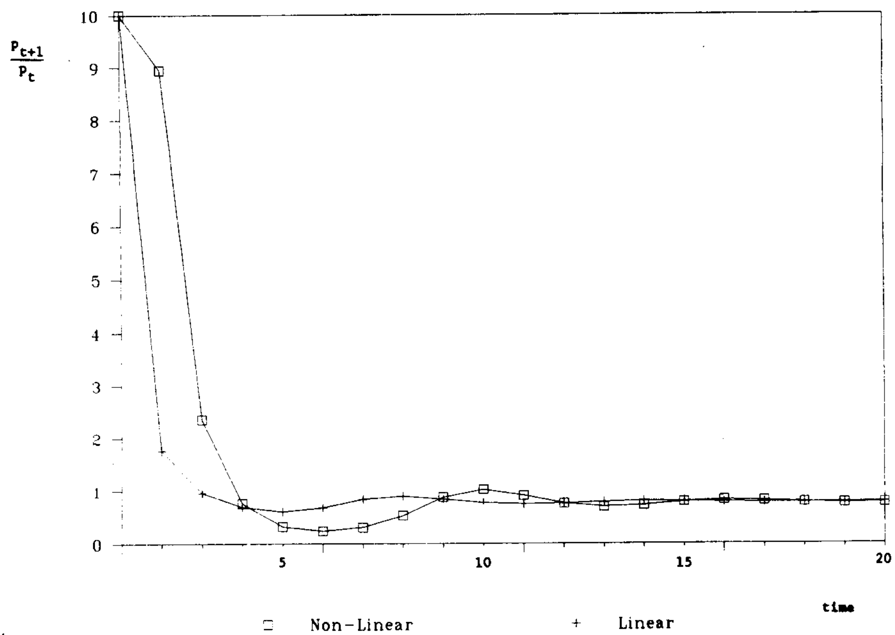
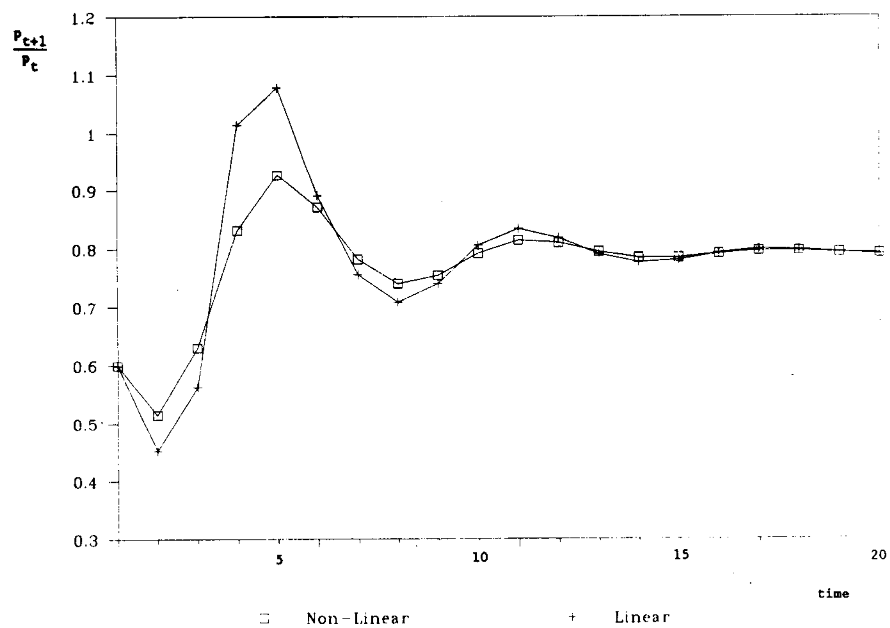

$\frac{p_{t+1}}{p_t}$
10.0
5.0
2.0
1.0
0.5
0.2
0.1
A
B
C
5
10
15
20
time

Fig. 2



$\frac{p_{t+1}}{p_t}$
10
9
8
7
6
5
4
3
2
1
0
5
10
15
20
time
Non-Linear
+ Linear

Fig. 3A



$\frac{p_{t+1}}{p_t}$
1.2
1.1
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
5
10
15
20
time
Non-Linear
+ Linear

Fig. 3B

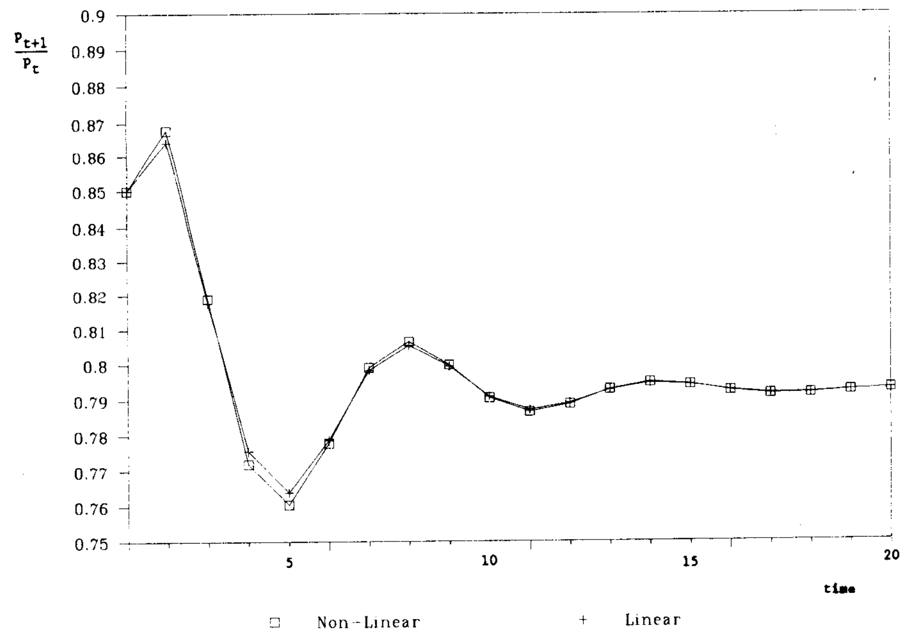


Fig. 3C

mation becomes better the closer the path starts to the steady state. Fig. 3A shows that when $p_2/p_1=10.00$, which is grossly different from the steady-state value of 0.79, the actual equilibrium is tracked relatively poorly by the linear approximation. The approximation indicates $p_3/p_2 \approx 1.77$, while in fact $p_3/p_2=8.95$. As a result of this gross error the linear approximation is thrown out of phase with the actual equilibrium. Fig. 3B shows that in the intermediate case when $p_2/p_1=0.60$ there is no phase error in the approximation although the discrepancy between the approximation $p_4/p_3 \approx 1.08$ and the actual value $p_4/p_3=0.93$ is substantial in economic terms. Fig. 3C shows that when $p_2/p_1=0.85$, which is still substantially above the steady-state price ratio of 0.79, the approximation error is nevertheless negligible. In all three cases the approximation has the same qualitative features as the actual equilibrium.

To construct an example of a two-dimensional continuum of Pareto-efficient equilibria with valued fiat money that converge to the same efficient steady state, let us consider the case where $m_{-1}=-0.29215$ and $m_0=0.45295$, so that the initial old people are in debt to the middle-aged. Here it can be checked that $(p_1,p_2,p_3,p_4)=(1,1,1,1)$ satisfies (2.6) and (2.7). In this case the excess demands of generations -1 and 0 are not homogeneous and we are not permitted a price normalization: money itself serves as numeraire. We can now choose $p_1=1+\varepsilon_1$ and $p_2=1+\varepsilon_2$ for any ε_1 and ε_2 small enough and use (2.8) to solve for an equilibrium price sequence that converges to one where $p_{t+1}/p_t=1$.

To see that the equilibria in the example above are Pareto efficient, observe that those converging to the steady state where $\beta=0.79247$ all assign finite value to the aggregate endowment, and so the standard proof of the first welfare theorem due to Debreu (1954) applies. Those that converge to the steady state where $\beta=1$ satisfy the more general conditions for efficiency developed by Balasko and Shell (1980) and Burke (1987).

The relative price indeterminacy exhibited in the monetary case does not depend on one of the consumers coming into the first year with negative fiat money. An alternative example has $m_{-1}=0$, $m_0=0.16080$, and the hump for initial middle-aged people increased so that $w_2^0=8.55977$ and $w_3^0=1$. Then $(p_1,p_2,p_3,p_4)=(1,1,1,1)$ satisfies the equilibrium conditions in the first two periods. Again there is a two-dimensional indeterminacy. [Setting $m_{-1}=m_0=0$ does not, however, result in equilibrium conditions that are satisfied by $p_t=(0.79247)^{t-1}$.]

Notice that our example also has steady states of the more familiar sort: any equilibrium that converges to the steady state where $\beta=0.17563$ is determinate. Any equilibrium that converges to $\beta=44.63396$ and has no fiat money is also determinate. There is a one-dimensional manifold of paths that converge to this steady state if there is fiat money, however.

Before presenting our next example, let us digress briefly to consider the possibility that instead of indeterminacy, there is instability: a steady state may not be approached at all by equilibria that start nearby. To illustrate this, consider the previous economy running backwards in time. Consequently, $a=2$ (the reciprocal of 0.5) and $(w_1,w_2,w_3)=(1,12,3)$ [rather than $(3,12,1)$]. Of course, $b=-3$. (The reversing of time in this economy is not meant to have any economic meaning; it simply provides us with an example with quantitative features that can easily be derived from those of the previous example. Other examples with more realistic parameters but the same qualitative features as this example are easy to construct.) This economy also has four steady states with β's and other roots that are reciprocals of those given above. Here the steady state where $\beta=1$ is unstable: there are no equilibria that converge to it unless, by pure chance, $(1,1,1,1)$ satisfies the equilibrium conditions in the first two periods. The steady state where $\beta=1.26188=(0.79247)^{-1}$ is also unstable for price paths with no fiat money. There are, however, locally unique equilibria with nonzero fiat money that converge to this steady state. Notice that this economy not only exhibits instability at the real steady state where $\beta=1.26188$ but also at the nominal steady state where $\beta=1$. This also follows by inverting the moduli of the eigenvalues.

Turning to the final example, we consider the possibility of a two-

Table 4

Steady states	β	Other roots		
1	0.07204	1	0.01617	0.55359
2	1	1.15172	$0.24628\pm$	$0.69144i$
3	1.15697	1	$0.29816\pm$	$0.81591i$
4	72.70100	1	1.761675	−183.42115

dimensional indeterminacy near an inefficient real steady state, where $\beta>1$. Suppose that $a=0.5$ and $b=-3$ as previously, but that $(w_1,w_2,w_3)=(3,15,1)$ rather than $(3,12,1)$. Suppose, in other words, that the life-cycle hump is a little higher. This economy has four steady states with β's and other roots as listed in table 4. The modulus of the pair of complex conjugates at the steady states where $\beta=1$ is 0.7399; where $\beta=1.15697$ it is 0.86862.

The interesting steady state is $\beta=1.15697$. There is a one-dimensional manifold of equilibria that converge to this steady state if there is no fiat money. If we allow the possibility that fiat money has value, however, then there is actually a two-dimensional manifold of equilibria.

An essential feature of these above examples is that they are robust: we can perturb slightly the parameters, including the functional forms, of demand by any or all consumers (including the initial old) and still have an economy with equilibria that have the same qualitative features. Indeed, we have chosen initial old consumers so that the steady-state prices satisfy the equilibrium conditions in the first two periods only to make it easy to verify that there are prices that satisfy these equilibrium conditions and also converge to the steady state.

Indeterminacy of the interest rate in a three-period-lived model may also be interpreted as indeterminacy of relative prices in a two-consumer, two-good, two-period-lived model. Indeed, Balasko, Cass and Shell (1980) show that the former is a special case of the latter. Suppose that consumers $h=1,2$ in generation t solve the utility maximization problem

$$\max \sum_{j=1}^{2}\sum_{i=1}^{2}\alpha_{ij}^{h}(c_{ij}^{\gamma_h}-1)/\gamma_h$$

s.t.

$$\sum_{j=1}^{2}\sum_{i=1}^{2} p_{it+j-1}c_{ij} \leqq \sum_{j=1}^{2}\sum_{i=1}^{2} p_{it+j-1}w_{ij}^{h}, \tag{4.3}$$

where, for example, c_{ij} is the consumption of good i in period $t+j-1$. If we set $\gamma_1=\gamma_2=b$, set $\alpha_{11}^1=\alpha_{21}^2=1$, $\alpha_{21}^1=\alpha_{12}^2=a$, $\alpha_{12}^1=\alpha_{22}^2=a^2$, and $\alpha_{11}^2=\alpha_{22}^1=0$, and similarly set w_{ij}^h, then this model is formally the same as the three-period-lived model that we have considered. In this interpretation the indeterminacy of the initial interest rate becomes indeterminacy of the initial relative prices of the two goods. The main reason for using the three-period-lived model for examples is to keep the specification as simple as possible: while the two-period-lived model needs eighteen parameters to specify it, of which fifteen are not subject to normalization, our simple three-period-lived model needs only five, of which four are not subject to normalization. It is still the case, however, that any small perturbation in the parameters of the two-period-lived model results in an economy with equilibria that have the same qualitative features as do the examples that we have presented.

5. The truncated model

There is a close relationship between models with infinite time horizons and models with long, but finite, time horizons. On one hand, models with infinite horizons are primarily interesting insofar as they provide insights into the properties of finite-horizon models. On the other hand, to approximate the equilibria of an infinite-horizon model on a computer we would have to truncate the model after a finite number of periods.

The example with the equilibria we have calculated in the previous section has, as we have observed, the maximum possible dimension of indeterminacy. We can, therefore, calculate equilibria simply by choosing arbitrary values for initial prices (close enough to the steady state) and then solve the equilibrium conditions forward to find an exact equilibrium. Suppose, however, that we had a model with indeterminate equilibria but with less than the maximum possible dimension of indeterminacy. The values for initial prices would have to be chosen to lie in some lower dimensional manifold. Although we know that the best linear approximation to this stable manifold is the stable subspace of the linearized system, calculating points on this manifold exactly is, except in very special cases, an impossible task. We would therefore need to truncate the model and calculate approximate equilibria.

One way to truncate the model at period T would be to fix the expectations of what prices would be in periods $T+1$ and $T+2$. We could, for example, require that $p_{T+2}=\beta p_{T+1}=\beta^2 p_T$ in the terminal equilibrium conditions:

$$x_3(p_{T-3},p_{T-2},p_{T-1})+x_2(p_{T-2},p_{T-1},p_T)+x_1(p_{T-1},p_T,p_{T+1})=0, \quad (5.1)$$

$$x_3(p_{T-2},p_{T-1},p_T)+x_2(p_{T-1},p_T,p_{T+1})+x_1(p_T,p_{T+1},p_{T+2})=0. \quad (5.2)$$

[See Auerbach and Kotlikoff (1987) for an example of this approach.]

Another, sometimes equivalent, way to truncate the model would be to specify terminal young generations $T-1$ and T analogous to the initial old generations -1 and 0. The terminal equilibrium conditions then become

$$x_3(p_{T-3},p_{T-2},p_{T-1})+x_2(p_{T-2},p_{T-1},p_T)+x_1^{T-1}(p_{T-1},p_T,m_{T-1})=0, \tag{5.3}$$

$$x_3(p_{T-2},p_{T-1},p_T)+x_2^{T-1}(p_{T-1},p_T,m_{T-1})+x_1^T(p_T,m_T)=0. \tag{5.4}$$

The equilibria of this model are equilibria of a finite economy with transfer payments. For an equilibrium to exist, it is necessary that $m_{-1}+m_0+m_{T-1}+m_T=0$. If this condition holds, but the individual transfers are not zero, then changes in the price level result in changes in real transfer payments. We would expect this model to have a one-dimensional continuum of equilibria indexed by the level of real transfer payments. For any fixed level of real transfer payments, the equilibrium conditions involve a finite number of equations and the same finite number of unknowns. We would therefore expect this model to generically have determinate equilibria.

Solving for equilibria of the truncated model is a two-point boundary value problem: there are four initial values for prices; there are two restrictions implied by the initial conditions and two restrictions implied by the terminal conditions. In practice, there are several ways to calculate equilibria.

Auerbach and Kotlikoff (1987) use a nonlinear Gauss–Seidel method. [See Fair and Taylor (1983) for another application of this general method.] They start by guessing a solution, the steady state for instance. They then solve the model going forward using this guess as expectations for future variables. After they are done, they use the calculated solution as a new guess and repeat the process. They stop when, and if, the calculated solution agrees with the previous guess. This method may be limited in its applicability, however: Laitner (1988) has shown that each of the twelve examples studied by Auerbach and Kotlikoff are determinate (which is, of course, good news in terms of their comparative statics analysis). Kehoe and Levine (1989) further argue that the nonlinear Gauss–Seidel method that they use does not converge for examples that exhibit indeterminacy or instability.

Lipton, Poterba, Sachs and Summers (1982) propose a method widely used by engineers and physical scientists, called *multiple shooting*. They start by guessing the initial values for variables and solving for the resulting price path in much the same way as we have done in the previous section. They then adjust these initial values until the price path satisfies the terminal conditions. Unless the equilibria are completely indeterminate, most paths diverge very rapidly. Because of this the algorithm is very numerically unstable. They therefore propose dividing the time period into segments. They then guess terminal conditions for each segment, solve for a solution over each segment, adjust the terminal conditions, and then repeat. As Press, Flannery, Teukolsky and Vetterling. (1986) point out, however, shooting methods do not work well in situations where the value of the largest modulus of an eigenvalue is much greater than β.

Perhaps the easiest way to solve for an equilibrium might be some variant of Newton's method to solve the whole system simultaneously. To be sure, there is a large number of equations and unknowns. The Jacobian matrix of the excess demands, however, although very large, is a very special matrix: only the prices $(p_{t-2}, p_{t-1}, p_t, p_{t+1}, p_{t+2})$ matter in determining the excess demand in period t. Inverting this matrix, the principal step in each iteration of Newton method, could take advantage of its band-diagonal structure. There may be numerical problems in inverting this matrix when there is indeterminacy or instability, however. In such cases, Kehoe and Levine (1989) argue that the Jacobian matrix is nearly singular. They present an alternative method for computing equilibria in such cases based on the standard proof of the local stable manifold theorem, which relies on a contraction mapping [see Irwin (1980)].

If the truncation date T is large enough, then an equilibrium of the truncated model serves as a good approximation to an equilibrium of the actual model, at least in the early periods. In fact, the usual proof of the existence of equilibrium for the infinite-horizon overlapping generations model depends on this property of the truncated model [see, for example, Balasko, Cass and Shell (1980)].

How does indeterminacy in the infinite model manifest itself in the truncated model? To answer this question, let us consider an infinite model with a continuum of equilibria that converge to the same steady state β. Choose two price paths in that continuum $(\bar{p}_1, \bar{p}_2, \ldots)$ and $(\hat{p}_1, \hat{p}_2, \ldots)$. For T large enough, both $(\bar{p}_{T+1}/\bar{p}_T, \bar{p}_{T+2}/\bar{p}_{T+1})$ and $(\hat{p}_{T+1}/\hat{p}_T, \hat{p}_{T+2}/\hat{p}_{T+1})$ are very close to (β, β) and consequently each other. Imposing the terminal conditions (5.1) with $p_{T+1} = (\bar{p}_{T+1}/\bar{p}_T)p_T$ and $p_{T+2} = (\bar{p}_{T+2}/\bar{p}_{T+1})p_{T+1}$, we generate $(\bar{p}_1, \bar{p}_2, \ldots, \bar{p}_T)$ as an equilibrium; with $\hat{p}_{T+1}$ and $\hat{p}_{T+2}$, we generate $(\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_T)$. No matter how large the difference between $\bar{p}_1$ and $\hat{p}_1$ there exists a T large enough so that $\bar{p}_T$ and $\hat{p}_T$ are arbitrarily close. Indeterminacy in the infinite model can, therefore, be seen to manifest itself as sensitivity to terminal conditions in the truncated model, sensitivity that becomes more and more acute as the truncation date becomes larger and larger.

Notice that in a model where $m=0$, we need only fix the terminal ratio p_{T+2}/p_{T+1}: homogeneity allows us to set $p_{T+2}=1$, and fixing the ratio p_{T+2}/p_{T+1} fixes p_{T+1}. The equilibrium conditions now constitute a non-homogeneous system of T equations in the T unknowns $p_1, p_2, \ldots, p_T$. In a

Table 5

Terminal rate ($t=20$)		Initial rate ($t=1$)	
1.1615%	(0.79)	0.00%	(1.00)
1.1675%	(0.79)	0.81%	(0.85)
1.1739%	(0.79)	1.80%	(0.70)
1.1743%	(0.79)	3.53%	(0.50)
1.1779%	(0.79)	2.59%	(0.60)
1.1871%	(0.79)	−7.73%	(5.00)

Note: Numbers in parentheses are p_{t+1}/p_t.

model where $m \neq 0$, however, we need to fix the levels of both p_{T+1} and p_{T+2}.

To illustrate this point, we refer back to the 'realistic' example with a 3.5 percent subjective discount rate, a 0.25 intertemporal elasticity of substitution, an endowment pattern of (3,12,1), no money, and an initial middle-aged endowment pattern of (8.27,1). (This example has been illustrated in figs. 1–3.) If each period is 20 years long, the 20th period corresponds to a date 400 years from the initial period. If we restrict attention to equilibria without valued fiat money, we can dispense with one of the two terminal conditions as discussed above. Consequently, we can truncate the model by specifying a single terminal interest rate. Using the equilibrium conditions (2.6)–(2.8), we can calculate the relationship between the terminal annual interest rate and the uniquely determined initial annual interest rate (table 5). As this table makes clear, in an economy with equilibria that are indeterminate in the infinite horizon, trivial changes in the anticipated interest rate in the distant future have an astonishing impact on initial equilibrium interest rates.

Another way to view this problem of indeterminacy, credited by Calvo (1978) to Rolf Mantel, is to consider the difference equation

$$x_3(p_{T-s-2}, p_{T-s-1}, p_{T-s}) + x_2(p_{T-s-1}, p_{T-s}, p_{T-s+1}) + x_1(p_{T-s}, p_{T-s+1}, p_{T-s+2}) = 0, \tag{5.5}$$

$s=1,2,\ldots$, that runs backwards from the terminal conditions. The roots of the polynomial that correspond to the linearized version of this equation are the reciprocals of the roots of (3.5). If the original system has too many stable roots, the backwards system has too many unstable roots: small changes in the terminal conditions cause large changes in prices as the path moves backwards over time.

How does instability in the infinite model manifest itself in the truncated model? Here we seem to be faced with a dilemma: we know that, if we truncate the model by requiring that $p_{T+2} = \beta p_{T+1} = \beta^2 p_T$, we can compute an approximate equilibrium for the infinite-horizon model. We also know, however, that it is extremely unlikely for the infinite-horizon model to have an equilibrium where p_{T+1} and p_{T+2} are close to these values. The solution to this dilemma lies in the nature of the approximation. We only know that the equilibria of the truncated model are close to the equilibria of the actual model in early periods; later they may diverge sharply. To get a good approximation to the equilibria of an infinite model near an unstable steady state for a prespecified number of periods, we may have to choose a very large truncation date.

As we would expect, the problems of indeterminacy and instability represent two sides of the same coin. Indeterminacy manifests itself as sensitivity to terminal conditions. The larger the truncation date, the more sensitive prices early in the price path are to terminal conditions. Later prices, however, which all converge to the steady state, are relatively insensitive. Instability, in contrast, manifests itself as a need for a very large truncation date. The larger the truncation date, the less sensitive prices early in the price path are to terminal conditions. Later prices, however, may diverge sharply from equilibrium prices for the actual model. As we have mentioned, however, it is not clear that perfect foresight is a sensible hypothesis in such situations.

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