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On characterizing equilibria of economies with externalities and taxes as solutions to optimization problems*

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Summary. We characterize equilibria of general equilibrium models with externalities and taxes as solutions to optimization problems. This characterization is similar to Negishi's characterization of equilibria of economies without externalities or taxes as solutions to social planning problems. It is often useful for computing equilibria or deriving their properties. Frequently, however, finding the optimization problem that a particular equilibrium solves is difficult. This is especially true in economies with multiple equilibria. In a dynamic economy with externalities or taxes there may be a robust continuum of equilibria even if there is a representative consumer. This indeterminacy of equilibria is closely related to that in overlapping generations economies.

1 Introduction

In this paper we characterize equilibria of general equilibrium models with externalities and taxes as solutions to optimization problems. Our general framework consists of a concave maximization problem that depends on a vector of parameters. For any vector of parameters the maximization problem has a unique solution. A set of side conditions relates this solution to the values of the parameters. An equilibrium is a vector of parameters that is consistent with the solution to the maximization problem that they imply. Formally, this characterization is similar to Negishi's (1960) characterization of equilibria of

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economies with heterogeneous consumers, but no taxes or externalities, as solutions to social planning problems. Negishi's characterization has an interpretation in terms of Pareto efficiency. Although no such normative interpretation is available in economies with externalities and taxes, this sort of characterization is often useful for computing equilibria or deriving their properties.

This topic has been the subject of much recent research: Abel and Blanchard (1983), Becker (1985), Danthine and Donaldson (1986), and Judd (1987) have investigated situations in which equilibria can be characterized as solutions to maximization problems without side constraints. Romer (1986) has applied the general approach to economies with externalities (earlier applications to economies with externalities include Arrow 1962, Brock 1977, and Kydland and Prescott 1977); Braun (1988), L.-J. Chang (1988), Jones and Manuelli (1988), Kehoe and Levine (1985b), and McGrattan (1988) have applied it to economies with taxes; and Ginsburgh and van der Heyden (1988) have applied it to economies with institutionally fixed prices.

Another application of this idea is to characterize the steady states of dynamic programming problems as solutions to a finite dimensional optimization problem with side conditions. See Koopmans (1971), Hansen and Koopmans (1972), Feinstein and Luenberger (1981), and Becker and Foias (1986). In this literature the general approach is sometimes called the implicit programming approach because the optimization problem to be solved depends on parameters that, in turn, depend on solutions to the problem.

There are, of course, formal similarities among the problems that can be solved using this general approach. Brock (1973), for example, demonstrates the formal equivalence between solving for the equilibrium of a single agent, static economy with a tax distortion and solving for the steady state of a multisector optimal growth model. This equivalence can be useful for translating results dealing with such matters as existence and uniqueness of solutions for one sort of problem into analogous results for another.

This paper presents a unifying framework for these various applications. It also focuses on the difficulties that arise because the parameters of the maximization problem are endogenously determined. In general, the parameters that satisfy the side conditions are found by solving a fixed point problem. This fixed point problem may have multiple solutions. Foster and Sonnenschein (1970) provide a static example in which a tax distortion causes multiplicity of equilibria. Howitt and McAfee (1988) and Spear (1988) provide dynamic examples in which externalities cause multiplicity, in fact, continua of equilibria. We fit these sorts of examples into our framework and provide a dynamic tax example with a continuum of equilibria.

One interesting aspect of examples with multiplicity of equilibria is that they illustrate the importance of the fixed point problem involving the parameters and side conditions: no single strictly concave maximization problem can have multiple equilibria as solutions. Another interesting aspect, emphasized by Spear (1988), is that a continuum of equilibria in a dynamic economy is usually associated with the possibility of sunspot equilibria. These are equilibria in which extrinsic uncertainty, uncertainty that does not directly affect the utility functions, endowments, or production technology, affects the equilibrium solely because agents expect it to. Indeed, Woodford's (1986) method for constructing sunspot equilibria in overlapping generations economies with continua of deterministic equilibria can easily be extended to the examples in this paper.

A final aspect of multiplicity of equilibria in economies with externalities and taxes that we investigate is its relation to multiplicity in overlapping generations economies. We demonstrate that equilibria of overlapping generations economies that solve a social planning problem, and are therefore Pareto efficient, fit into our framework. Furthermore, there are robust examples with continua of such equilibria. It is also possible to extend our framework to other dynamic economies with continua of equilibria, such as Woodford's (1988) version of the Lucas–Stokey (1987) cash-in-advance model, although we do not do so here. (In fact, Brock 1975a has applied this type of framework to a money-in-the-utility-function model with a continuum of equilibria.)

One aspect of multiplicity of equilibria in these economies that we do not investigate is its relevance for applied work. The examples that we present here are not intended to be especially realistic. Even in the overlapping generations framework the question of the empirical relevancy of multiplicity has not been resolved; Kehoe and Levine (1990) show that a continuum of equilibria can occur in a robust example with plausible parameters. Laitner (1984, 1990), however, shows that it does not occur in a large number of interesting examples.

In the next section we outline the general framework and relate it to Negishi's (1960) characterization of equilibria in economies without taxes or externalities. In Sect. 3 we illustrate how this framework can be applied to models with externalities; in Sect. 4 we do the same for economies with taxes. We demonstrate the possibility of multiplicity of equilibria in our examples in Sect. 5. We explore the similarities between multiplicity of equilibria in dynamic economies with distortions and that in overlapping generations economies in Sect. 6. Finally, in Sect. 7 we discuss the advantages and limitations of our characterization of equilibria as solutions to optimization problems.

2 General framework

To illustrate how equilibria can be characterized as solutions to optimization problems, what the value of this characterization is, and what its limitations are, we study a series of simple examples. All of the examples fit into a general framework: Equilibria solve the problem of choosing the vector x to solve

$$\max w(x, z)$$

subject to

$$x \in \Gamma(z).$$

Here w is a distorted social welfare function; x includes individual demand and supply vectors by consumers and firms; and z is a vector of parameters that describes taxes and distortions and, in an economy with heterogeneous consumers, includes weights on individual utilities. The constraint set $\Gamma(z)$ describes social feasibility. The maximum theorem says that, if w is continuous in x and z and concave in x and if Γ is a convex-valued, continuous correspondence in x, then the solution $x(z)$ and the associated vector of Lagrange multipliers $p(z)$ are convex-valued correspondences that vary upper-hemi-continuously with z. If w is strictly concave in x, then $x(z)$ is a continuous, single-valued function; if w is continuously differentiable, then $p(z)$ is also a continuous, single-valued function. Typically, $p(z)$ can be interpreted as a vector of prices. In equilibrium the parameters z are endogenous in that they depend on the solution to the optimization problem: in addition to solving the above optimization problem, an equilibrium (x, p, z) must satisfy additional conditions,

$$\psi(x, p, z) = 0,$$

where the vector $\psi(x, p, z)$ has the same dimension as z. Using the optimization problem to define x and p as functions of z, this becomes a system of equations in z, $\psi(x(z), p(z), z) = 0$.

To illustrate the applicability of this framework in a familiar context, consider a pure-exchange economy with two consumers, two goods, and no externalities or taxes. Each consumer chooses the consumption plan (c_1^i, c_2^i) to solve

$$\max u^i(c_1^i, c_2^i)$$

subject to

$$p_1 c_1^i + p_2 c_2^i \leqq p_1 w_1^i + p_2 w_2^i$$
$$c_j^i \geqq 0.$$

Here u^i is a continuously differentiable, strictly concave, and monotonically increasing utility function, p_1 and p_2 are prices, and w_1^i and w_2^i are endowments.

The Kuhn–Tucker theorem says that the necessary and sufficient conditions for a solution to this problem are that there exists a nonnegative Lagrange multiplier λ_i such that

$$u_j^i(c_1^i, c_2^i) - \lambda_i p_j \leqq 0, \quad = 0 \quad \text{if} \quad c_j^i > 0, \quad j = 1, 2$$
$$p_1 w_1^i + p_2 w_2^i - p_1 c_1^i - p_2 c_2^i \geqq 0, \quad = 0 \quad \text{if} \quad \lambda_i > 0.$$

Here u_j^i is the partial derivative of u^i with respect to c_j^i. An equilibrium of this model is a vector $(p_1, p_2, c_1^1, c_2^1, c_1^2, c_2^2)$ such that each consumer maximizes utility taking prices as given and

$$c_j^1 + c_j^2 - w_j^1 - w_j^2 \leqq 0, \quad = 0 \quad \text{if} \quad p_j > 0, \quad j = 1, 2.$$

Consider now the problem of choosing the allocation $(c_1^1, c_2^1, c_1^2, c_2^2)$ to solve the social planning problem

$$\max \alpha_1 u^1(c_1^1, c_2^1) + \alpha_2 u^2(c_1^2, c_2^2)$$

subject to

$$c_j^1 + c_j^2 \leqq w_j^1, \quad j = 1, 2$$
$$c_j^i \geqq 0$$

where α_1 and α_2 are nonnegative welfare weights. The necessary and sufficient conditions for a solution to this problem are that there exist nonnegative Lagrange multipliers p_1 and p_2 such that

$$\alpha_1 u^i_j(c^i_1, c^i_2) - p_j \leqq 0, \quad = 0 \quad \text{if} \quad c^i_j > 0, \quad i = 1,2; \quad j = 1,2$$
$$w^1_j + w^2_j - c^1_j - c^2_j \geqq 0, \quad = 0 \quad \text{if} \quad p_j > 0, \quad j = 1,2.$$

Notice if we set c^i_j and p_j equal to their equilibrium values and set $\alpha_1 = 1/\lambda_i$, then an equilibrium solves this social planning problem. This is, of course, the first welfare theorem: a competitive equilibrium is Pareto efficient. Notice too that any Pareto efficient allocation and associated Lagrange multipliers satisfy all of the equilibrium conditions except, possibly, the individual budget constraints. This is the second welfare theorem: any Pareto efficient allocation can be implemented as a competitive equilibrium with transfer payments.

Negishi's (1960) approach to characterizing equilibria of this economy is to consider savings functions that indicate the extent to which the allocation associated with the weights (α_1, α_2) violates the budget constraints:

$$s_i(\alpha_1, \alpha_2) = \sum_{j=1}^{2} p_j(\alpha_1, \alpha_2)(w^i_j - c^i_j(\alpha_1, \alpha_2)).$$

Setting these savings functions equal to zero corresponds to the equilibrium conditions $\psi(c, p, \alpha) = 0$ in the general formulation. Rather than calculating equilibria by solving for prices such that the excess demand functions are equal to zero, Negishi proposes solving for welfare weights such that the savings functions are equal to zero. Indeed, saving functions have many similarities with excess demand functions: since the savings functions are homogeneous of degree one and sum to zero, the functions $s_i(\alpha_1, \alpha_2)/\alpha_i$ have the same formal properties as do the excess demand functions of a (different) pure-exchange model with two goods. To calculate an equilibrium, we need only to solve $s_1(\alpha, 1 - \alpha) = 0$. This approach is frequently easier than solving for prices such that the excess demand functions are equal to zero, especially in economies with more goods than consumers.

3 Economies with externalities

We first apply our framework to economies with externalities. We consider two examples, the first static, the second dynamic. Although these models are not especially realistic, the general method extends to move complicated models and, indeed, has been used to study growth with spillover effects. (See Romer 1988 for a discussion of these models.) To keep the presentation simple, we retain the representative consumer framework here and in the subsequent two sections. As in the previous section, we could readily use the method of Negishi to incorporate heterogeneity of consumers.

Example 1. Consider a static economy with a congestion externality. There is a continuum of identical consumers. Let $u(c, x)$ denote the utility function of the representative consumer where c is his consumption of the single produced good and x is his consumption of leisure. We assume that u is continuously differentiable, strictly concave, and monotonically increasing. (These assumptions are, of course, much stronger than they need be; they are made only to facilitate the presentation.) The consumer is endowed with one unit of labor, which can either be consumed as leisure or used to produce output of the consumption good.

Output of the consumption good is produced by an infinite number of identical firms. Because there is a congestion externality, the production function for the representative firm, $f(\ell, L)$, depends not only on the amount of labor input, ℓ, but also on the average amount of labor input throughout the economy, L. We assume that $f(\ell, L)$ is continuously differentiable and satisfies $f_1 > 0$, $f_{11} < 0$, and $f_2 < 0$.

Faced by prices (p, w) of output and labor and by an average level of profits from production π, the consumer chooses (c, x) to solve

$$\max u(c, x)$$

subject to

$$\begin{aligned} pc + wx &\leqq w + \pi \\ c, x &\geqq 0. \end{aligned}$$

The necessary and sufficient conditions for a solution to this problem are that there exists a nonnegative Lagrange multiplier λ such that

$$\begin{aligned} u_1(c, x) - \lambda p &\leqq 0, \quad =0 \quad \text{if} \quad c > 0 \\ u_2(c, x) - \lambda w &\leqq 0, \quad =0 \quad \text{if} \quad x > 0 \\ w + \pi - pc - wx &\geqq 0, \quad =0 \quad \text{if} \quad \lambda > 0. \end{aligned}$$

Faced by prices (p, w) and an average level of labor input L elsewhere in the economy, the firm chooses ℓ to solve

$$\max pf(\ell, L) - w\ell.$$

The necessary and sufficient conditions for a solution to this problem are

$$pf_1(\ell, L) - w \leqq 0, \quad =0 \quad \text{if} \quad \ell > 0.$$

An equilibrium of this economy is a vector $(p, w, \pi, L, c, x, \ell)$ such that the consumer maximizes utility taking p, w, and π as given: the firm maximizes profits taking p, w, and L as given; and

$$\begin{aligned} \pi &= pf(\ell, L) - w\ell \\ L &= \ell \\ c - f(\ell, L) &\leqq 0, \quad =0 \quad \text{if} \quad p > 0 \\ x + \ell - 1 &\leqq 0, \quad =0 \quad \text{if} \quad w > 0. \end{aligned}$$

Equilibria of this economy are not, of course, Pareto efficient in general. Consider, however, the problem of choosing c, x, and ℓ to solve the "Pareto" problem

$$\max u(c, x)$$

subject to

$$\begin{aligned} c &\leqq f(\ell, L) \\ x + \ell &\leqq 1 \\ c, x, \ell &\geqq 0. \end{aligned}$$

Notice that L is taken as a given parameter. Associated with each level of L there is a different problem and a different associated solution. The necessary and sufficient conditions for a solution to this problem are that there exist nonnegative Lagrange multipliers p and w that

$$\begin{aligned} u_1(c,x)-p &\leqq 0, \quad =0 \quad \text{if} \quad c>0 \\ u_2(c,x)-w &\leqq 0, \quad =0 \quad \text{if} \quad x>0 \\ pf_1(\ell,L)-w &\leqq 0, \quad =0 \quad \text{if} \quad \ell>0 \\ f(\ell,L)-c &\geqq 0, \quad =0 \quad \text{if} \quad p>0 \\ 1-x-\ell &\geqq 0, \quad =0 \quad \text{if} \quad w>0. \end{aligned}$$

If we normalize prices so that $\lambda = 1$ and set $\pi = pf(\ell, L) - w\ell$, then a solution to this problem satisfies all of the equilibrium conditions except, possibly, $\ell = L$.

Finding solutions to the side condition $\ell = L$ is equivalent to finding equilibria of this economy in the same way that finding solutions to the side condition $s_1(\alpha, 1-\alpha) = 0$ is equivalent to finding equilibria of the exchange economy of the previous section. For any level of average labor input L, let $\ell(L)$ be the labor demand in the solution to the above problem. The strict concavity of u and f imply that ℓ is a continuous function of L. Since $0 \leqq \ell(L) \leqq 1, \ell(L)$ has at least one fixed point as in Fig. 1.

Example 2. We next consider a dynamic economy with a consumption externality. The infinitely lived representative consumer chooses the consumption sequence $c_0, c_1, \ldots,$ to solve

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to

$$\sum_{t=0}^{\infty} p_t c_t \leqq \sum_{t=0}^{\infty} y_t$$

$$c_t \geqq 0.$$

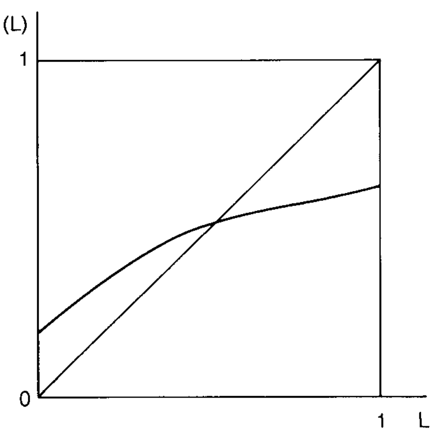

Fig. 1. See text

Here $\beta, 0 < \beta < 1$, is a discount factor; $u(c_t)$ is continuously differentiable, strictly concave, and monotonically increasing; $p_0, p_1, \ldots,$ is a sequence of prices; and $y_0, y_1, \ldots,$ is a sequence of incomes.

The production sector has an externality in that production of the consumption goods depends not only on inputs of capital k_t and of fixed amount of labor, but also on average consumption C_t. Although this model is not intended to be realistic, we could imagine such an externality arising because of diseases that are contagious at low levels of consumption. We assume that the production function $f(k_t, C_t)$ is continuously differentiable and satisfies $f_1 > 0$, $f_{11} < 0$, and $f_{12} > 0$. Implicitly, there is a production function $F(k_t, \ell_t, C_t)$ that exhibits constant returns in k_t and ℓ_t. If the rate of depreciation of capital is δ,

$$f(k_t, C_t) = F(k_t, 1, C_t) + (1 - \delta)k_t.$$

The necessary and sufficient conditions for a solution to this problem are that there exists a nonnegative Lagrange multiplier λ such that

$$\beta^t u'(c_t) - \lambda p_t \leqq 0, \quad = 0 \quad \text{if} \quad c_t > 0, \quad t = 0, 1, \ldots$$
$$\sum_{t=0}^{\infty} y_t - \sum_{t=0}^{\infty} p_t c_t \geqq 0, \quad = 0 \quad \text{if} \quad \lambda > 0$$
$$\lim_{t \to \infty} p_t c_t = 0.$$

Faced by the sequences $p_0, p_1, \ldots,$ and $C_0, C_1, \ldots,$ the representative firm chooses $k_0, k_1, \ldots,$ to solve

$$\max \sum_{t=0}^{\infty} (p_t f(k_t, C_t) - p_t k_{t+1}) - r_0 k_0.$$

The necessary and sufficient conditions for a solution are

$$p_0 f_1(k_0, C_0) - r_0 \leqq 0, \quad = 0 \quad \text{if} \quad k_0 > 0$$
$$p_{t+1} f_1(k_{t+1}, C_{t+1}) - p_t \leqq 0, \quad = 0 \quad \text{if} \quad k_{t+1} > 0, \quad t = 0, 1, \ldots$$
$$\lim_{t \to \infty} p_t k_{t+1} = 0.$$

An equilibrium is a sequence of the form $(p_t, y_t, C_t, c_t, k_t)$ and a price of the initial capital stock r_0 that satisfy the above maximization conditions for the consumer and firm and the additional conditions

$$y_t = p_t f(k_t, C_t), \quad t = 0, 1, \ldots$$
$$c_t + k_{t+1} - f(k_t, C_t) \leqq 0, \quad = 0 \quad \text{if} \quad p_t > 0, \quad t = 0, 1, \ldots$$
$$k_0 - \bar{k}_0 \leqq 0, \quad = 0 \quad \text{if} \quad r_0 > 0$$
$$C_t = c_t.$$

The "Pareto" problem for this economy is

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to

$$\begin{aligned} c_t + k_{t+1} &\leqq f(k_t, C_t), \quad t = 1, 2, \dots \\ k_0 &\leqq \bar{k}_0 \\ c_t, k_t &\geqq 0. \end{aligned}$$

Once again a solution to this problem satisfies all of the equilibrium conditions except, possibly, the side conditions $C_t = c_t$. For any sequence $C_0, C_1, \dots$, the strict concavity of u and f imply that this optimization problem has a unique solution. Solving for an equilibrium is, however, significantly more difficult than in the previous example: we need to find a fixed point of an infinite dimensional function,

$$C_t = c_t(C_0, C_1, \dots), \quad t = 0, 1, \dots$$

It is possible to impose conditions on f that would ensure that a fixed point of this function, and therefore an equilibrium, exists. Since the primary focus of this paper is on the characterization, rather than the existence, of equilibria, however, we shall not pursue this issue.

4 Economies with taxes

We next apply our framework to economies with tax distortions. Unlike the externality examples, here we modify the utility function instead of the production function to account for the distortion. As with the externality examples, we retain the representative consumer framework and consider both a static and a dynamic example.

Example 3. Consider a static economy with two consumption goods and a representative consumer who faces an *ad valorem* tax on purchases of one of the goods. He is endowed with a fixed amount of labor that he inelastically supplies to a constant-returns production technology to produce the two consumption goods. The production possibility set for this economy is defined by the inequalities

$$\begin{aligned} c_1 + ac_2 &\leqq b \\ c_1, c_2 &\geqq 0, \end{aligned}$$

where b is proportional to the fixed labor input.

The problem faced by the consumer is

$$\max u(c_1, c_2)$$

subject to

$$\begin{aligned} p_1(1+\tau)c_1 + p_2 c_2 &\leqq y \\ c_1, c_2 &\geqq 0. \end{aligned}$$

Here, of course, c_1 and c_2 are the levels of consumption of the goods, p_1 and p_2 the prices, y the consumer's income, and τ the *ad valorem* tax on purchases of the first good. We assume that all tax revenues are rebated in lump sum form to the representative consumer. The necessary and sufficient conditions for a solution to this problem are that there exists a nonnegative Lagrange multiplier λ such that

$$u_1(c_1, c_2) - \lambda p_1(1+\tau) \leqq 0, \quad = 0 \quad \text{if} \quad c_1 > 0$$
$$u_2(c_1, c_2) - \lambda p_2 \leqq 0, \quad = 0 \quad \text{if} \quad c_2 > 0$$
$$y - p_1(1+\tau)c_1 - p_2 c_2 \geqq 0, \quad = 0 \quad \text{if} \quad \lambda > 0.$$

An equilibrium of this economy is a vector (p_1, p_2, y, c_1, c_2) such that the consumer maximizes utility taking p_1, p_2, and y as given; c_1 and c_2 are technologically feasible; and

$$p_2 = ap_1$$
$$y = p_1 c_1 + p_2 c_2 + \tau p_1 c_1.$$

These final two conditions ensure that there are zero profits in production and that all tax revenues are rebated in lump sum form to the consumer.

Consider, for a fixed nonnegative constant z, the "Pareto" problem of choosing c_1 and c_2 to solve

$$\max u(c_1, c_2) - zc_1$$

subject to

$$c_1 + ac_2 \leqq b$$
$$c_1, c_2 \geqq 0.$$

The necessary and sufficient conditions for a solution to this problem are

$$u_1(c_1, c_2) - z - p \leqq 0, \quad = 0 \quad \text{if} \quad c_1 > 0$$
$$u_2(c_1, c_2) - ap \leqq 0, \quad = 0 \quad \text{if} \quad c_2 > 0$$
$$b - c_1 - ac_2 \geqq 0, \quad = 0 \quad \text{if} \quad p > 0.$$

If we set $p_1 = p, p_2 = ap$, and $\lambda = 1$, the a solution to this problem is an equilibrium if the side condition $z = \tau p(z)$ is satisfied.

Example 4. We next apply our framework to a dynamic economy with distortionary taxes. Consider an economy in which the representative consumer derives utility not only from consumption, but from investment. Capital here can be thought of as human capital; the consumer values education not only for its enhancement of future production possibilities but for its own sake. Kurtz (1968) provides an alternative interpretation of a similar model. Purchases of the consumption good are subject to an *ad valorem* tax. Once again, all tax revenues are rebated in lump sum form to the consumer. Unlike the dynamic economy in the previous section where, to keep the accounting simple, we think of the firm as holding all the capital after period 0, consumers here purchase the investment good in one period and then sell it to the firm in the next. The representative firm chooses the sequence $k_0, k_1, \ldots,$ to solve the problem

$$\max \sum_{t=0}^{\infty} (p_t f(k_t) - r_t k_t).$$

The consumer chooses the sequence $(c_0, x_0), (c_1, x_1), \ldots$ to solve the problem

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t, x_t)$$

subject to

$$\sum_{t=0}^{\infty} (p_t(1+\tau)c_t + q_t x_t) \leqq \sum_{t=0}^{\infty} (y_t + r_{t+1} x_t)$$

$$c_t, x_t \geqq 0.$$

Here x_t is purchases of the investment good by the consumer in period t, q_t is the price paid, and r_{t+1} is the price paid by the firm for the same good in period $t+1$.

An equilibrium is a sequence $(p_t, q_t, r_t, y_t, c_t, x_t, k_t)$ such that the consumer maximizes utility taking p_t, q_t, y_t, and r_t as given; the firm maximizes profits taking p_t and r_t as given; and

$$\begin{aligned}
y_0 &= p_0 f(k_0) + \tau p_0 c_0 \\
y_t &= p_t f(k_t) - r_t k_t + \tau p_t c_t, \quad t = 1, 2, \ldots \\
c_t + a x_t - f(k_t) &\leqq 0, \quad = 0 \text{ if } p_t > 0, \quad t = 0, 1, \ldots \\
q_t &= a p_t, \quad t = 0, 1, \ldots \\
k_0 - k_0 &\leqq 0, \quad = 0 \text{ if } r_0 > 0 \\
k_t - x_{t-1} &\leqq 0, \quad = 0 \text{ if } r_t > 0, \quad t = 1, 2, \ldots
\end{aligned}$$

The "Pareto" problem for this economy is

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t, k_{t+1}) - z_t c_t$$

subject to

$$\begin{aligned}
c_t + a k_{t+1} &\leqq f(k_t), \quad t = 1, 2, \ldots \\
k_0 &\leqq \bar{k}_0 \\
c_t, k_{t+1} &\geqq 0.
\end{aligned}$$

The necessary and sufficient conditions for a solution to this problem are

$$\begin{aligned}
\beta^t u_1(c_t, k_{t+1}) - z_t - p_t &\leqq 0, \quad = 0 \text{ if } c_t > 0, \quad t = 0, 1, \ldots \\
p_0 f'(k_0) - r_0 &\leqq 0, \quad = 0 \text{ if } k_0 > 0 \\
\beta^t u_2(c_t, k_{t+1}) - a p_t + p_{t+1} f'(k_{t+1}) &\leqq 0, \quad = 0 \text{ if } k_{t+1} > 0, \quad t = 0, 1, \ldots \\
f(k_t) - c_t - a k_{t+1} &\geqq 0, \quad = 0 \text{ if } p_t > 0, \quad t = 0, 1, \ldots \\
\bar{k}_0 - k_0 &\geqq 0, \quad = 0 \text{ if } r_0 > 0 \\
\lim_{t \to \infty} p_t k_{t+1} &= 0.
\end{aligned}$$

A solution to this problem satisfies all of the equilibrium conditions except, possibly, the side conditions $z_t = \tau p_t$. Equilibria of this economy are solutions to the infinite dimensional fixed point problem

$$z_t = \tau p_t(z_0, z_1, \ldots), \quad t = 0, 1, \ldots$$

5 Multiplicity of equilibria

As we have indicated, extra endogenous variables and side conditions necessitate an analysis different than that for pure optimization. In the static case, despite the representative consumer, a finite number of multiple equilibria are possible and, indeed, may even be Pareto ranked. In the dynamic case, the infinity of extra variables and side conditions can lead to a robust continuum of equilibria near a steady state; as Woodford (1986) has shown, this is necessary and sufficient for the existence of sunspot equilibria. Here we illustrate these points in the four examples considered in the previous two cases. We first examine the two tax economies in some detail and then extend the analysis to the externality economies.

Example 3 revisited. Even though it has a representative consumer and a production technology that fixes producer prices, our static tax economy can have multiple equilibria, as has been pointed out by Foster and Sonnenschein (1970). To illustrate this point and its implications for our approach, consider a numerical example: The consumer has a (concave) quadratic utility function

$$u(c_1, c_2) = 22c_1 + 27c_2 - 1/2(3c_1^2 + 8c_1c_2 + 6c_2^2).$$

The production possibility set is

$$\begin{aligned} c_1 + 3c_2 &\leqq 6 \\ c_1, c_2 &\geqq 0. \end{aligned}$$

The first order conditions for the "Pareto" problem are

$$\begin{aligned} (22 - 3c_1 - 4c_2) - z - p \leqq 0, &\quad = 0 \quad \text{if} \quad c_1 > 0 \\ (27 - 4c_1 - 6c_2) - 3p \leqq 0, &\quad = 0 \quad \text{if} \quad c_2 > 0 \\ 6 - c_1 - 3c_2 \geqq 0, &\quad = 0 \quad \text{if} \quad p > 0. \end{aligned}$$

Solving this system of inequalities, we obtain

$$p(z) = \begin{cases} 4 - z & \text{if} \quad z \leqq 3 \\ (2z - 3)/3 & \text{if} \quad 3 \leqq z \leqq 9. \\ 5 & \text{if} \quad z \geqq 9 \end{cases}$$

Suppose that the *ad valorem* tax rate on consumption of the first good is 200 percent; in other words, $\tau = 2$. The equation $z = 2p(z)$ has three solutions, each of which corresponds to an equilibrium

z	p	c_1	c_2
8/3	4/3	6	0
6	3	3	1
10	5	0	2

These three equilibria are depicted in Fig. 2. Notice that the equilibria are Pareto ranked: $u(6, 0) > u(3, 1) > u(0, 2)$.

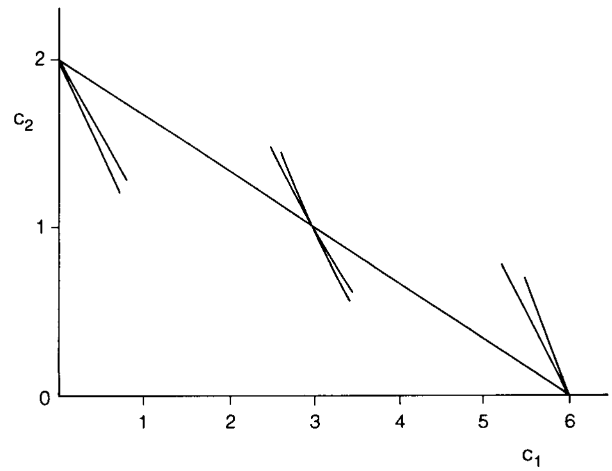

Fig. 2. See text

This example provides a stark illustration of the fundamental importance of the fixed point problem that must be solved to find an equilibrium. Even though there is a representative consumer and, for any fixed z, the "Pareto" problem is a concave optimization problem with a unique solution, the fixed point problem has three isolated solutions. It is impossible for there to be any single concave optimization problem that has each of these three equilibria as solutions.

Example 4 revisited. The problem of multiplicity of equilibria is more acute in dynamic economies than it is in static economies. It is easy, for example, to construct a robust example in which our dynamic tax economy has an infinite number of equlibria. To do so, let us impose the side conditions on the first order conditions for the "Pareto" problem. If c_t, k_t, and p_t are all strictly positive, we can combine the three corresponding first order conditions to obtain the second order difference equation

$$(1+\tau)u_2(f(k_t)-ak_{t+1},k_{t+1})-au_1(f(k_t)-ak_{t+1},k_{t+1})$$
$$+\beta u_i(f(k_{t+1})-ak_{t+2},k_{t+2})f'(k_{t+1})=0.$$

This difference equation requires two initial conditions. The value of k_0 is given. How much freedom is there in choosing k_1? To answer this question, we can linearize this difference equation around a stationary solution $\hat{k}$. (To guarantee the existence of such a solution we would have to impose additional restrictions on f; the example that we shall construct, however, has such a stationary solution.) The local stable manifold theorem from the theory of dynamical systems says that, in nondegenerate cases, what is true of the linearization is true of the nonlinear system itself in some open neighborhood of the stationary solution (see, for example, Irwin 1980, Chapter 6).

To study the qualitative behavior of the linearization, we compute the roots of its characteristic polynomial, in this case a quadratic. If both of the two roots of this quadratic are less than one in modulus, then all values of k_0 and k_1 lead to convergence to the stationary solution. If one root is less than one in modulus and the other is greater, then there is a one dimensional affine set of values of k_0 and k_1 that lead to convergence to the stationary solution. If neither root is less than one in modulus, then all solutions to the difference equation diverge from the stationary solution unless $k_0 = k_1 = \hat{k}$.

The local stable manifold theorem says that, if both roots are less than one in modulus, then the stable manifold of the nonlinear system – the set of values (k_0, k_1) that lead to convergence to $\hat{k}$ – is an open neighborhood of $(\hat{k}, \hat{k})$ in R^2; if one root is greater than one and one root less, then it is a one dimensional manifold that contains $(\hat{k}, \hat{k})$ and whose best linear approximation near $(\hat{k}, \hat{k})$ is the stable set of the linearized system; and if neither is less than one, then it is the single point $(\hat{k}, \hat{k})$. It is only when one of the roots is exactly equal to one in modulus that behavior of the linearized system provides no guide to behavior of the nonlinear system.

Since k_0 is given, two roots less than one in modulus implies that there is a continuum of equilibria indexed by the choice of k_1: Given k_0 and k_1, we can compute the sequence $k_2, k_3, \ldots$, using the difference equation. The values of the other variables can then be computed using the equilibrium conditions. A steady state of this economy is a vector $(\hat{p}, \hat{q}, \hat{r}, \hat{y}, \hat{c}, \hat{x}, \hat{k})$ such that the sequence

$$(p_t, q_t, r_t, y_t, c_t, x_t, k_t) = (\beta^t \hat{p}, \beta^t \hat{q}, \beta^t \hat{r}, \beta^t \hat{y}, \hat{c}, \hat{x}, \hat{k})$$

satisfies all of the equilibrium conditions except possibly $\hat{k} = \bar{k}_0$. Given a stationary solution $\hat{k}$ to the difference equation, we can easily use the equilibrium conditions to compute the steady state values of p, q, r, y, c, and x. Studying solutions to the difference equation that converge to $\hat{k}$ is equivalent, therefore, to studying equilibria that converge to the corresponding steady state.

The economy may have other equilibria that diverge from the steady state, or it may have equilibria in which some c_t, p_t, or r_0 is equal to 0, in which case the first order conditions are not all equalities. If, however, the steady state values of $\hat{c}, \hat{p}$, and $\hat{r}$ are all strictly positive, then two roots of the characteristic quadratic being less than one in modulus ensures a continuum of equilibria if $\bar{k}_0$ is close enough to $\hat{k}$. Notice that it is impossible for both roots to be less than one in modulus in a model whose equilibria solve a concave optimization problem without endogenous parameters: since there is a unique equilibrium, the roots exhibit either saddle-point splitting, one less than one in modulus and the other greater, or instability, both greater than one.

The relevant linearized difference equation has the form

$$\alpha_2(k_{t+2} - \hat{k}) + \alpha_1(k_{t+1} - \hat{k}) + \alpha_0(k_t - \hat{k}) = 0$$

where

$$\alpha_0 = ((1 + \tau)u_{21} - au_{11})f'$$
$$\alpha_1 = -a(1+\tau)u_{21} + (1+\tau)u_{22} + a^2 u_{11} - au_{12} + \beta u_{11} f'^2 + \beta u_1 f''$$
$$\alpha_2 = \beta(u_{12} - au_{11})f'.$$

The characteristic equation is

$$\alpha_2 \lambda^2 + \alpha_1 \lambda + \alpha_0 = 0.$$

The two roots can easily be computed using the quadratic formula. Suppose that we normalize so that $\alpha_2 \geqq 0$ (by reversing the signs of the coefficients necessary). Then the necessary and sufficient conditions for both to be less than one in modulus are

$$\alpha_2 - \alpha_0 > 0$$
$$\alpha_2 + \alpha_1 + \alpha_0 > 0$$
$$\alpha_2 - \alpha_1 + \alpha_0 > 0.$$

We can construct a numerical example with a continuum of equilibria by modifying Example 3 considered previously. Suppose that the momentary utility function is

$$u(c_t, k_{t+1}) = 22c_t + 27k_{t+1} - 1/2(3c_t^2 + 8c_t k_{t+1} + 6k_{t+1}^2).$$

The *ad valorem* tax rate is again $\tau = 2$. Suppose that $f(1) = 6 + \beta f'(1)$ and $a = 3 + \beta f'(1)$. Then $\hat{k} = 1$ is a stationary solution to the difference equation that corresponds to the equilibrium $(c_1, c_2) = (3, 1)$ of the previous model. At the steady state $(\hat{p}, \hat{q}, \hat{r}, \hat{y}, \hat{c}, \hat{x}, \hat{k}) = (3, 9 + 3\beta f', 3f', 36, 3, 1, 1)$,

$$u_1 = 9$$
$$u_{11} = -3$$
$$u_{12} = u_{21} = -4$$
$$u_{22} = -6.$$

Consequently,

$$\alpha_0 = -3f' + 3\beta f'^2$$
$$\alpha_1 = 3 - 2\beta f' - 3\beta f'^2 - 3\beta^2 f'^2 + 9\beta f''$$
$$\alpha_2 = 5\beta f' + 3\beta^2 f'^2.$$

For a wide range of choices for β, f', and f'', both roots are less than one in modulus. Suppose, for example, that $\beta = 2/3$, $f' = 1$, and $f'' = -1/9$. In this case, the production function could be the (concave) quadratic

$$f(k) = 20/3 + (1)(k-1) - 1/2(1/9)(k-1)^2$$
$$= (101 + 20k - k^2)/18.$$

The characteristic equation is

$$(14/3)\lambda^2 - (7/3)\lambda - 1 = 0,$$

whose two solutions are

$$\lambda = \frac{7 \pm \sqrt{217}}{28} = 0.7761, -0.2761.$$

The multiplicity of equilibria of this example arises because equilibria are solutions to a fixed point problem involving an infinite number of equations and unknowns. Such fixed point problems often exhibit infinite numbers of solutions. See Kehoe, Levine, Mas-Colell, and Zame (1989) for a discussion of the mathematical issues involved. Notice once again that the optimization problem that any equilibrium solves is concave. In particular, for every fixed sequence of parameters $z_0, z_1, \ldots$, there is a unique solution.

Using the techniques of differential topology, we could show that each of the two tax examples that we have studied is a regular economy, and that any small perturbations to the parameters of functional forms of these examples give rise to economies whose equilibria have the same qualitative features. These examples of multiplicity are, therefore, robust. See Mas-Colell (1985) for a discussion of regularity in static economies without externalities or taxes and Kehoe, Levine, and Romer (1990) for a discussion of regularity in dynamic economies; Kehoe (1985) shows how these concepts can be applied to tax economies.

There are also robust examples with unique equilibria, however. In particular, suppose that tax distortions are small. In either economy, Example 3 or Example 4, we know that there is a unique equilibrium if $\tau = 0$. Except for degenerate cases, the economy where $\tau = 0$ is a regular economy. Consequently, for any τ small enough the economy also has a unique equilibrium.

To illustrate this point consider again the static tax economy, Example 3. We can compute the equilibria for different values of τ by solving the equation $z = \tau p(z)$. For $0 \leqq \tau < 9/5$ there is a unique equilibrium where $z = 4\tau/(1+\tau)$, $p = 4/(1+\tau)$, and $(c_1, c_2) = (6, 0)$. For $9/5 < \tau < 3$ there are three equilibria:

z	p	c_1	c_2
$4\tau/(1+r)$	$4/(1+\tau)$	6	0
$3\tau(2\tau-3)$	$3/(2\tau-3)$	$(15\tau-27)/(2\tau-3)$	$(3-\tau)/(2\tau-3)$
5τ	5	0	2

For $\tau > 3$ there is a unique equilibrium where $z = 5\tau$, $p = 5$, and $(c_1, c_2) = (0, 2)$.

For every value of τ except 9/5 and 3 the economy is regular. It is only at these two critical economies that the qualitative characteristics of the set of equilibria changes. Every economy with $\tau < 9/5$, for example, has a unique equilibrium where $(c_1, c_2) = (6, 0)$. At the critical economy with $\tau = 9/5$, however, a new equilibrium appears at $(c_1, c_2) = (2, 0)$. See Fig. 3.

Characterizing all of the equilibria of our dynamic tax economy as τ varies is a far more difficult task. It is, however, a simple matter to argue that, if τ is small enough, there cannot be a continuum of equilibria that converge to any steady state. The necessary and sufficient condition for the two roots of the characteristic quadratic to be both less than one in the modulus can be combined to yield the necessary condition

$$|\alpha_2| - |\alpha_0| > 0$$

where $\alpha_2 = \beta(u_{12} - au_{11})f'$ and $\alpha_0 = ((1+\tau)u_{12} - au_{11})f'$. If $\tau = 0$, then this expression becomes

$$(\beta - 1)|u_{12} - au_{11}| > 0,$$

which is impossible since $\beta < 1$. It is easy to check that this condition cannot be satisfied if τ is small enough.

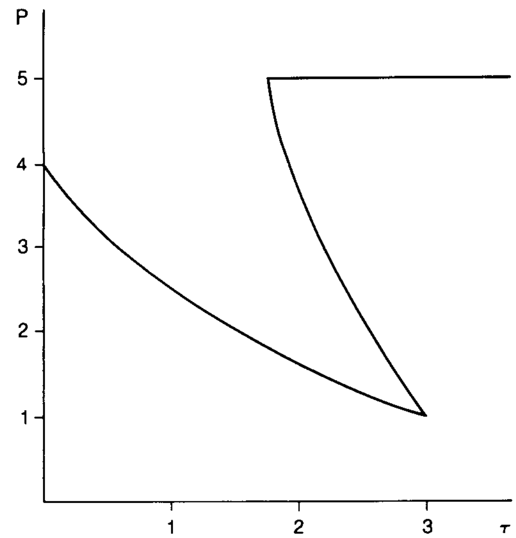

Fig. 3. See text

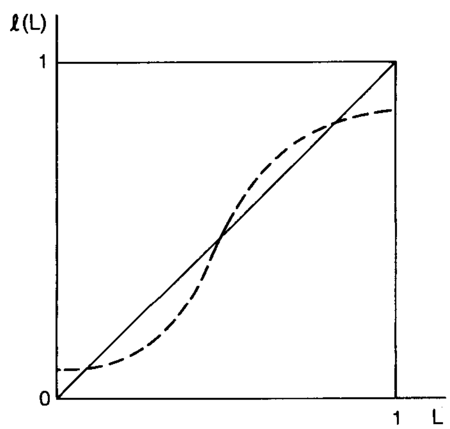

Fig. 4. See text

Example 1 revisited. Like distortionary taxes, externalities can create multiplicity of equilibria in economies that would otherwise have unique equilibria. Consider again our static externality economy. As we have seen, an equilibrium is a solution to the fixed point problem $L = \ell(L)$, where $\ell(L)$ is implicitly defined by the first order conditions of the "Pareto" problem. At an interior solution these conditions can be combined to yield

$$u_1(f(\ell, L), 1-\ell) f_1(\ell, L) - u_2(f(\ell, L), 1-\ell) = 0.$$

Suppose that $L = \ell(L), 0 < L < 1$, is a solution to the fixed point problem and that, at the equilibrium where $c = f(L, L)$, $x = 1 - L$, and $\ell = L$,

$$u_{11} f_1^2 - 2u_{12} f_1 + u_1 f_{11} + u_{22} \neq 0.$$

Then the implicit function theorem says that $\ell(L)$ is continuously differentiable in some open neighborhood of L and that

$$\ell'(L) = \frac{u_{11}f_1f_2 + u_1f_{12} - u_{12}f_2}{u_{11}f_1^2 - 2u_{12}f_1 + u_1f_{11} + u_{22}}.$$

Suppose now that the functions u and f are such that there is such a solution L where $\ell'(L) > 1$. (Such an example is, of course, easy to construct using quadratic functions as in the static tax example.) Then there are necessarily at least two more equilibria as Fig. 4 illustrates.

Once again, however, there is necessarily a unique equilibrium if the externality is small enough. If the externality is small, then f_2 and f_{12} are close to zero. In this case, $\ell'(L)$ is always close to 0. For there to be multiple equilibria, however, it is necessary that $\ell'(L) \geqq 1$ at some equilibrium.

Example 2 revisited. It is also easy to construct a dynamic economy that has a continuum of equilibrium because of an externality. Such models have been constructed by Howitt and McAfee (1988) and Spear (1988). Indeed, the dynamic externality economy presented earlier can have a continuum of equilibria. Suppose that the first order conditions of the "Pareto" problem are all satisfied with equality. Then we can reduce the problem of finding an equilibrium to the problem of finding solutions to the system of two first order difference equations

$$-u'(c_t) + \beta u'(c_{t+1})f_1(k_{t+1}, c_{t+1}) = 0$$
$$f(k_t, c_t) - k_{t+1} - c_t = 0.$$

Once again there is only one initial condition, $k_0 = \bar{k}_0$.

As a stationary solution $(\hat{c}, \hat{k})$, if one exists, this system can be linearized as

$$\begin{bmatrix} u'' + \beta u' f_{12} & \beta u' f_{11} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c_{t+1} - \hat{c} \\ k_{t+1} - \hat{k} \end{bmatrix} = \begin{bmatrix} u'' & 0 \\ f_2 - 1 & f_1 \end{bmatrix} \begin{bmatrix} c_t - \hat{c} \\ k_t - \hat{k} \end{bmatrix}.$$

We can choose u and f so that there is a stationary solution to this difference equation with a continuum of solutions that converge to it and satisfy $k_0 = \bar{k}_0$. To do so, we choose u and f so that the two eigenvalues of the matrix

$$\begin{bmatrix} u'' + \beta u' f_{12} & \beta u' f_{11} \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} u'' & 0 \\ f_2 - 1 & f_1 \end{bmatrix}$$

are both less than one in modulus. These eigenvalues are solutions to the characteristic equation

$$\alpha_2\lambda^2 + \alpha_1\lambda + \alpha_0 = 0$$

where

$$\alpha_0 = -u''/\beta$$
$$\alpha_1 = u''/\beta + u'f_{12} + u'' - \beta u' f_{11}(f_2 - 1)$$
$$\alpha_2 = -u'' - \beta u' f_{12}.$$

Suppose, for example, that $(\hat{c}, \hat{k}) = (2, 1)$, $\beta = 1/2$, $u'(2) = 8$, $u''(2) = -2$, $f_1(1, 2) = 2$, $f_2(1, 2) = 2$, $f_{11}(1, 2) = -2$, and $f_{12}(1, 2) = -3/4$. Then $\alpha_0 = 4$, $\alpha_1 = -4$, and $\alpha_2 = 5$.

In this case the two eigenvalues are $\lambda = 2/5 \pm 4/5i$, which have modulus $|\lambda| = 0.8944$. As with the dynamic tax example, we can choose quadratic functions u and f that satisfy these restrictions.

Notice that, for there to be a continuum of equilibria, we require that f_2 and f_{12} be significantly different from 0. If f_2 and f_{12} are both close to 0, in other words, if the externality is small, then the linearized system exhibits either instability or saddle-point splitting. This rules out a continuum of equilibria that converge to the steady state where $(\hat{c}, \hat{k}) = (2, 1)$.

6 Multiplicity of equilibria in overlapping generations models

Our discussion of the possibility of multiplicity in the dynamic economies with externalities and taxes is suggestive of similar discussions of indeterminacy of equilibria in overlapping generations economies (see, for example, Kehoe and Levine 1985a). In this section we examine the similarities between these two approaches. Just as multiplicity of equilibria can occur in the pure-exchange economy with two consumers and two goods in which all equilibria are Pareto efficient because of heterogeneity of consumers, indeterminacy can occur in an overlapping generations economy because of the infinite number of consumers and not because some equilibria are not Pareto efficient. This suggests that the continua of equilibria in the dynamic models presented earlier is due to the infinite number of welfare weights and side conditions and not because of Pareto inefficiency of equilibria. See Kehoe, Levine, Mas-Colell, and Zame (1989) for a discussion of the relationship between indeterminacy and equilibrium conditions with infinite numbers of equations and unknowns.

Example 5. To illustrate the possibility of a continuum of equilibria, each of which solves a social planning problem, in an overlapping generations economy, we consider the model presented by Kehoe and Levine (1990) in which there is a single consumer, who lives for three periods, in each period and a single good in each period. Such an example cannot be constructed with consumers who live for two periods and one good in each period; it is possible to interpret our example, however, as one in which there are two consumers who live for two periods in each generation and two goods in each period.

The consumer born in generation $t, t = 1, 2, \ldots$, chooses the consumption plan $(c_t^t, c_{t+1}^t, c_{t+2}^t)$ to solve

$$\max u(c_t^t, c_{t+1}^t, c_{t+2}^t)$$

subject to

$$p_t c_t^t + p_{t+1} c_{t+1}^t + p_{t+2} c_{t+2}^t \leqq p_t w_1 + p_{t+1} w_2 + p_{t+2} w_3$$
$$c_s^t \geqq 0, \quad s = t, \quad t+1, \quad t+2.$$

Here u is a continuously differentiable, strictly concave, monotonically increasing utility function and (w_1, w_2, w_3) is an endowment vector. Notice that, although this model is stationary in the sense that u and (w_1, w_2, w_3) do not change over time, it allows a great deal of heterogeneity of consumers: consumer 1, for example, has utility for, and endowments of, only goods 1, 2, and 3, while consumer 5 is concerned only with goods 5, 6, and 7. In addition to these consumers, there is a consumer who lives only in period 1 and 2. He solves

$$\max u^0(c_1^0, c_2^0)$$

subject to

$$p_1 c_1^0 + p_2 c_2^0 \leqq p_1 w_2^0 + p_2 w_3^0.$$

We could also allow for an additional consumer who lives only in period 1, but, since the only equilibria we consider here have no fiat money, such a consumer could do nothing more in equilibrium than consume his own endowment.

An equilibrium of this model is a price sequence $p_1, p_2, \ldots$, and an allocation $(c_1^0, c_2^0), (c_1^1, c_2^1, c_3^1), \ldots$, such that consumers maximize utility taking prices as given and the allocation satisfies

$$c_1^0 + c_1^1 - w_2^0 - w_1 \leqq 0, \quad = 0 \quad \text{if} \quad p_1 > 0$$

$$c_2^0 + c_2^1 + c_2^2 - w_3^0 - w_2 - w_1 \leqq 0, \quad = 0 \quad \text{if} \quad p_2 > 0$$

$$c_t^{t-2} + c_t^{t-1} + c_t^t - w_3 - w_2 - w_1 \leqq 0, \quad = 0 \quad \text{if} \quad p_t > 0, \quad t = 3, 4, \ldots.$$

Consider now the Pareto problem of choosing an allocation to maximize

$$\alpha_0 u^0(c_1^0, c_2^0) + \sum_{t=1}^{\infty} \alpha_t u(c_t^t, c_{t+1}^t, c_{t+2}^t)$$

subject to the above feasibility conditions. If an equilibrium satisfies the property

$$\sum_{t=1}^{\infty} p_t < \infty,$$

then it solves a maximization problem of this sort where

$$\alpha_t = 1/\lambda_t = p_t / u_1(c_t^t, c_{t+1}^t, c_{t+2}^t).$$

Overlapping generations economies typically have many equilibria that do not satisfy this property. Some are Pareto inefficient and some satisfy the more general Pareto efficiency criterion

$$\sum_{t=1}^{\infty} 1/p_t = \infty$$

(see Balasko and Shell 1980 and Burke 1986). In these cases the social welfare function would not converge at an equilibrium.

Actually, every equilibrium satisfies conditions that look like first order conditions for a Pareto problem. The difficulty is that the social welfare function may not converge. Using the overtaking criterion, we could extend our analysis to equilibria that satisfy the above general efficiency criterion. We could even use this approach to prove that this efficiency criterion is, under approximate conditions, necessary and sufficient for Pareto efficiency. We shall not pursue this issue, however.

Any solution to the Pareto problem satisfies all of the equilibrium conditions except the budget constraints. If we define savings functions as for the model with two consumers and two goods, equilibria correspond to a sequence of welfare weights such that

$$s_t(\alpha_0, \alpha_1, \ldots) = 0.$$

Again, this is a system with an infinite number of equations and unknowns. Kehoe and Levine (1990) present an example in which it has a continuum of solutions. In it consumer $t, t = 1, 2, \ldots$, has the utility function

$$u(c_t^t, c_{t+2}^t, c_{t+2}^t) = \sum_{i=0}^{2} \beta^i((c_{t+i}^t)^b - 1)/b$$

and consumer 0 has the utility function

$$u^0(c_1^0, c_2^0) = \sum_{i=1}^{2} \beta^i((c_i^0)^b - 1)/b.$$

They linearize the equilibrium conditions around a steady state and show that for $\beta = 1/2$, $b = -3$, and $(w_1, w_2, w_3) = (3, 12, 1)$, (w_2^0, w_3^0) can be chosen in many ways so that there is a continuum of equilibria, all of which converge to a steady state in which $p_t = (0.7925)^t$. Each of these equilibria is, of course, Pareto efficient and solves a social planning problem.

7 Advantages and limitations of the approach

To compute equilibria of general equilibrium models we must, in general, be able to solve fixed point problems. (See Uzawa 1962 and Scarf 1982 for a discussion of this issue.) The characterization of equilibria as solutions to optimization problems is useful in the computation of equilibria to the extent to which it is easy to find the optimization problem that an equilibrium solves. There is always a trivial optimization problem that an equilibrium $(\hat{x}, \hat{p})$ solves:

$$\max -(\|x - \hat{x}\|^2 + \|p - \hat{p}\|^2).$$

The only way we can find this problem, however, is to compute the equilibrium by some other means. This sort of characterization is obviously not very useful. Another point worth making about the optimization problems that we have considered is that they are all concave maximization problems, which have unique solutions that are easy to verify as solutions and relatively easy to compute. Any fixed point problem can be recast as an optimization problem,

$$\max -\|z - g(z)\|^2.$$

Because the objective function is not concave, however, this formulation is not very useful.

Sometimes equilibria of simple economies can be shown to solve optimization problems with few or no side conditions. Consider, for example, economies without externalities or taxes in which the consumers satisfy the aggregation conditions considered by Gorman (1953), where utility functions are all homothetic and identical but endowments arbitrary, and by Chipman (1974), where utility functions are again homothetic but possibly different and endowment vectors are proportional. In either case equilibria could, of course, be characterized as solutions to maximizing a weighted sum of individual utilities. There are alternative characterizations, however, that avoid the use of welfare weights that must be solved for side conditions.

Consider first the case where utility functions are identical. Let u be the homogeneous of degree one representation of the common utility function. Then the unique equilibrium of an economy with m consumers can be found by solving

$$\max u\left(\sum_{i=1}^{m} c^i\right)$$

subject to

$$\sum_{i=1}^{m} c^i - \sum_{i=1}^{m} w^i \in Y$$

where Y is the closed, convex production cone.

Consider now the case where utility functions are different, but endowments are all positive proportions θ_i, $\sum_{i=1}^{m} \theta_i = 1$, of the aggregate endowment vector w. (This condition ensures that the distribution of income is fixed.) Let u^i be the homogeneous of degree one representation of the utility function of consumer i. Then the unique equilibrium of this economy can be found by maximizing $\sum_{i=1}^{m} \theta_i \log u^i(c^i)$ subject to the feasibility constraints.

Similarly, there are tax economies in which the equilibrium solves a optimization problem without side conditions. Any such economy must, of course, have a unique equilibrium. The ease with which we can construct examples with multiple equilibria suggests that these models cannot be very general. Becker (1985) considers a model with a tax on capital holdings in which the representative consumer solves

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to

$$\sum_{t=0}^{\infty} p_t(c_t + k_{t+1}) \leqq \sum_{t=0}^{\infty} (y_t + (1-\tau)r_t k_t)$$

$$k_0 \leqq \bar{k}_0$$

$$c_t, k_t \geqq 0.$$

Here $y_t = p_t f(k_t) - r_t k_t + \tau r_t k_t$ is the consumer's labor income plus a lump sum rebate. Our approach would be to characterize the equilibrium as a solution to

$$\max \sum_{t=0}^{\infty} \beta^t (u(c_t) - z_t k_t)$$

subject to

$$c_t + k_{t+1} \leqq f(k_t), \quad t = 0, 1, \ldots$$
$$k_0 \leqq \bar{k}_0$$
$$c_t, k_t \geqq 0$$

along with the side conditions $z_t = \tau p_t f'(k_t)$. Becker, however, shows that an equilibrium also solves

$$\max \sum_{t=0}^{\infty} (\beta(1-\tau))^t u(c_t)$$

subject to

$$c_t + k_{t+1} \leqq f(k_t), \quad t = 0, 1, \ldots$$
$$k_0 \leqq \bar{k}_0$$
$$c_t, k_t \geqq 0.$$

Danthine and Donaldson (1986) extend Becker's analysis to economies that allow uncertainty. Judd (1987) shows that some similar continuous time tax economies also have equilibria that solve optimization problems without side conditions. He further argues that, although there are few cases in which equilibria of tax economies can be computed exactly by solving an optimization problem without side conditions, research in this area may be helpful in updating guesses in iterative methods for computing equilibria. Suppose, for example, that we fix z_t at its steady state value in a dynamic tax economy. We can then solve the optimization problem for this guess of z_t and use the solution to update z_t and so on. Kydland and Prescott (1977) and Whiteman (1983) have discussed using an algorithm of this sort to compute equilibria in economies with externalities. Further research is needed to see whether this algorithm has any advantages over alternatives.

A major attraction to being able to characterize an equilibrium of a model as a solution to an optimization problem without side conditions is that it assures us that the model has a unique equilibrium. Whether such an optimization problem exists for a given set of equilibrium conditions, regardless of whether or not it is easy to find, is an integrability question. Dechert (1978) provides necessary and sufficient conditions for discrete-time models, Brock (1975b) for deterministic continuous-time models, and F.-R. Chang (1988) for stochastic, continuous-time models. Unfortunately, although there is undoubtedly much room for research in this area, the integrability conditions appear to be either very difficult to check, or very restrictive.

Our approach is still attractive in situations where the number of side conditions is small. Consider, for example, an economy in which the government uses specific taxes, taxes such that the market price of a good is $p_j + \sigma_j$ rather than $p_j(1 + \tau_j)$, to finance a fixed vector of government purchases g. In a model with n goods and m consumers, we define the "Pareto" problem

$$\max \sum_{i=1}^{m} \alpha_i u^i(c^i) - \alpha_0 \sum_{j=1}^{n} \sigma_j \sum_{i=1}^{m} c_j^i$$

subject to

$$\sum_{i=1}^{m} (c^i - w^i) + g \in Y.$$

Here $\alpha_1, \ldots, \alpha_m$ are welfare weights, α_0 is a scale factor applied to the vector of specific taxes, $w^1, \ldots, w^m$ are endowment vectors, and Y is a production cone. An equilibrium corresponds to a vector $(\alpha_0, \alpha_1, \ldots, \alpha_m)$ that satisfies the side conditions

$$\alpha_0 \sum_{j=1}^{n} \sigma_j \sum_{i=1}^{m} c_j^i(\alpha) - \sum_{j=1}^{n} p_j(\alpha) g_j = 0$$

$$\sum_{j=1}^{n} [p_j(\alpha) w_j^i - (p_j(\alpha) + \alpha_0 \sigma_j) c_j^i(\alpha)] = 0, \quad i = 1, \ldots, m.$$

The first condition is the government budget constraint, while the other m conditions are the usual savings functions. Notice that there is a finite number, $m + 1$, of side constraints. This approach may be especially valuable in an optimal taxation framework where the government varies the vector σ to maximize social welfare: for any given vector σ, it needs to be able to compute the resulting equilibrium.

Our approach may also be useful for deriving properties of equilibria with taxes and distortions. Jones and Manuelli (1990) and Kehoe and Levine (1985b), for example, consider dynamic models that satisfy strong stationarity properties. Using techniques from the theory of dynamic programming, Jones and Manuelli are able to show that, because it solves an optimization problem, any equilibrium must satisfy conditions that make it easy to prove existence of equilibrium. Kehoe and Levine demonstrate that, in an economy with distortionary taxes, investment in heterogeneous capital goods must maximize a distorted social return function. Furthermore, prices of the investment goods are the partial derivatives of this function. Since the function is concave in the capital goods, Kehoe and Levine argue that, under certain conditions, this has empirical implications.

Undoubtedly, it would be possible to derive the results of Jones–Manuelli and Kehoe–Levine by examining the equilibrium conditions for the model without using the optimization characterization. The theory of dynamic programming provides a powerful set of tools, however, which can ease many burdens of proof.

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