Strategic Form Games

Finite Games

an N player game $i = 1 \ldots N$

$P(S)$ are probability measure on S

finite strategy spaces S_i

$\sigma_i \in \Sigma_i \equiv P(S_i)$ are mixed strategies

$s \in S \equiv \times_{i=1}^{N} S_i$ are the strategy profiles, $\sigma \in \Sigma \equiv \times_{i=1}^{N} \Sigma_i$

other useful notation $s_{-i} \in S_{-i} \equiv \times_{j \neq i} S_j$

$\sigma_{-i} \in \Sigma_{-i} \equiv \times_{j \neq i} \Sigma_j$

$u_i(s)$ payoff or utility

$u_i(\sigma) \equiv \sum_{s \in S} u_i(s) \prod_{j=1}^{N} \sigma_j(s_j)$ is expected utility

Dominance and Rationalizability

σ_i weakly (strongly) dominates σ'_i if

$$u_i(\sigma_i, s_{-i}) \geq (>) u_i(\sigma'_i, s_{-i}) \text{ with at least one strict}$$

Prisoner's Dilemma Game

	R	L
U	2,2	0,3
D	3,0	1,1

a unique dominant strategy equilibrium (D,L) this is Pareto dominated by (U,R) does it really occur??

Public Goods Experiment

Players randomly matched in pairs

May donate or keep a token

The token has a fixed commonly known public value of 15

It has a randomly drawn private value uniform on 10-20

V=private gain/public gain

So if the private value is 20 and you donate you lose 5, the other player gets 15; V= -1/3

If the private value is 10 and you donate you get 5 the other player gets 15; V=+1/3

Data from Levine/Palfrey, experiments conducted with caltech undergraduates

Based on Palfrey and Prisbey

V	donating a token
0.3	100%
0.2	92%
0.1	100%
0	83%
-0.1	55%
-0.2	13%
-0.3	20%

Second Price Auction

a single item is to be auctioned.

value to the seller is zero.

$i = 1, \ldots, N$ buyers

value $v_i > 0$ to buyer *i*.

each buyer submits a bid b_i

the item is sold to the highest bidder at the second highest bid

bidding your value weakly dominates

Iterated Dominance

example of iterated weak dominance

	L	R-l	R-r
U-u	-1,-1	2,0	1,1
U-d	-1,-1	1,-1	0,0
D	1,1	1,1	1,1

Eliminate U-d

Eliminate L

Eliminate D (or) Eliminate R-l

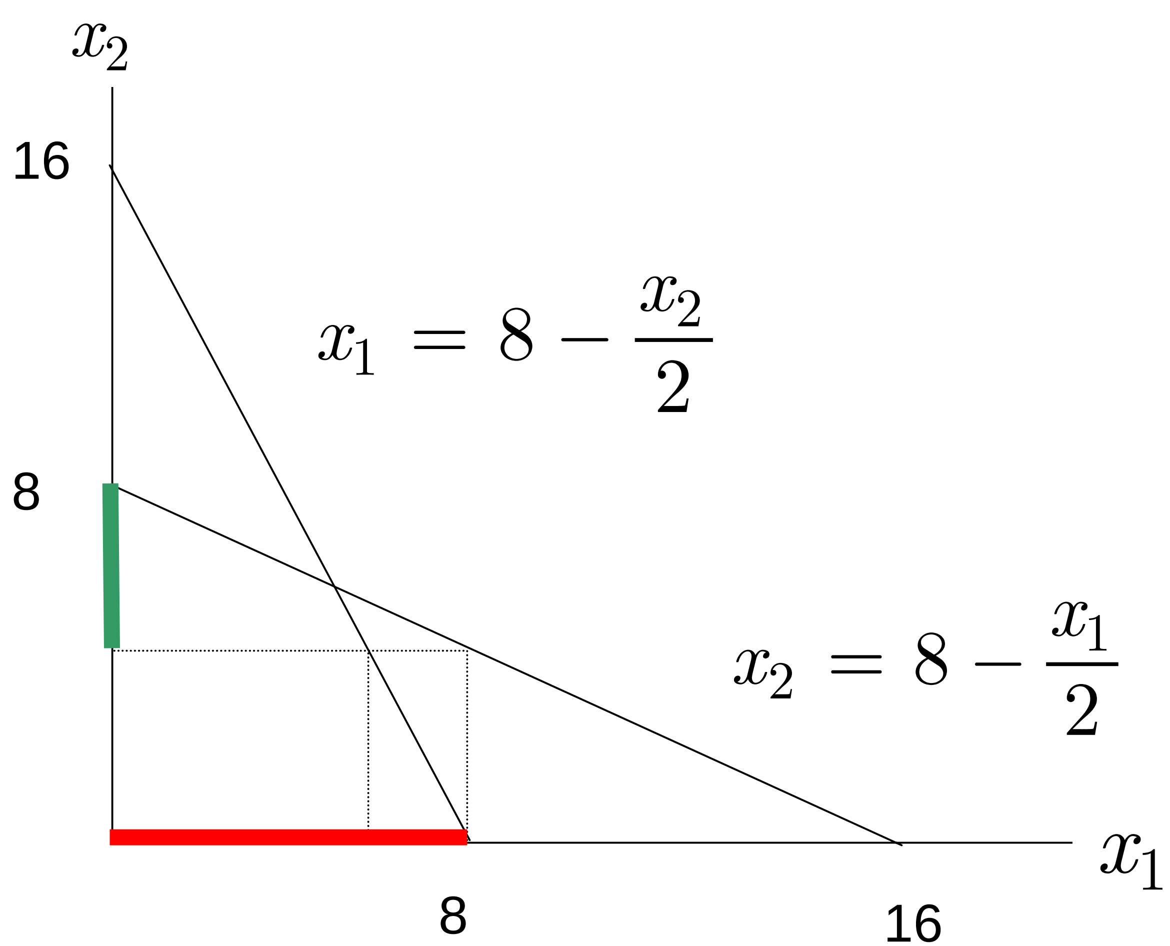
Eliminate R-I

Notice that there can be more than one answer for iterated weak dominance

Not for iterated strong dominance

Cournot Oligopoly

$\pi_i = [17 - (x_i + x_{-i})]x_i - x_i$; reaction function $x_i = 8 - \frac{x_{-i}}{2}$



Nash Equilibrium

Definition

players can anticipate on another's strategies

σ is a *Nash equilibrium* profile if for each $i \in 1, \ldots N$
$u_i(\sigma) = \max_{\sigma'_i} u_i(\sigma'_i, \sigma_{-i})$

Theorem: a Nash equilibrium exists in a finite game

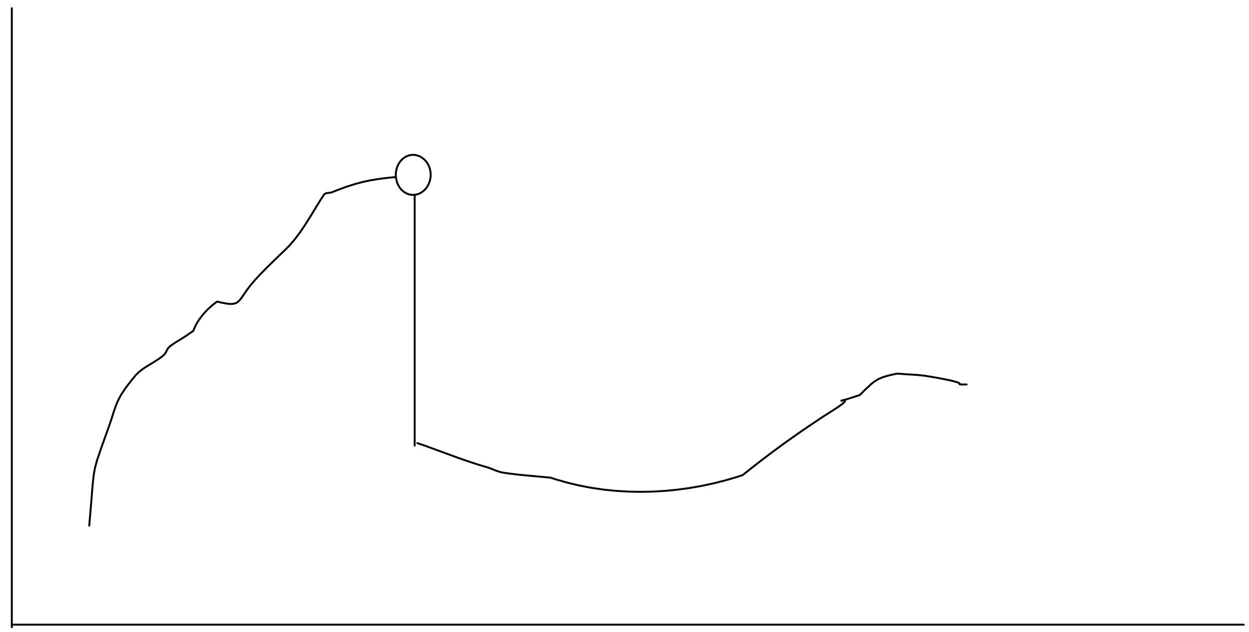
This theorem fails in pure strategies: consider matching pennies

Holmes and Moriarity

this is more or less why Kakutani's fixed point theorem was invented:

An upper hemi-continuous (UHC) convex valued correspondence B from a convex subset $\Sigma \subseteq \Re^n$ to itself has a fixed point $\sigma \in B(\sigma)$

A correspondence $B : \Sigma \rightrightarrows \Sigma$ is UHC means if $\sigma^n \to \sigma$ such that $b^n \in B(\sigma^n) \to b$ then $b \in B(\sigma)$.



Proof: Let $B_i(\sigma)$ be the set of best responses of i to σ_{-i}

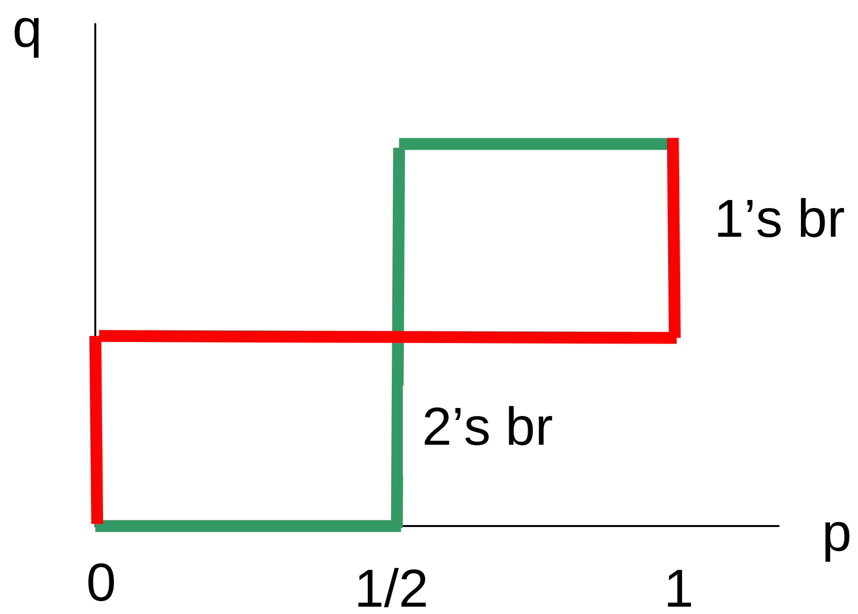
convex valued: convex combinations of a best response is a best response. Specifically, since you must be indifferent between all pure strategies played with positive probability, the best response set is the set of all convex combinations of the pure strategies that are best responses.

UHC: $b_i^n \in B_i(\sigma_{-i}^n) \to b_i$ means that $u^i(b_i^n, \sigma_{-i}^n) \geq u^i(\sigma_i, \sigma_{-i}^n)$. Suppose the converse that $b \notin B(\sigma)$. This means for some $\hat{\sigma}_i$ that $u^i(\hat{\sigma}_i, \sigma_{-i}) > u^i(b_i, \sigma_{-i})$. Since $\sigma_{-i}^n \to \sigma_{-i}$ for n sufficiently large, since u^i is continuous (multi-linear in fact) in σ_{-i}, $u^i(\hat{\sigma}_i, \sigma_{-i}^n) > u^i(b_i, \sigma_{-i}^n)$. Since $b_i^n \to b_i$, since u^i is continuous (linear in fact) in σ_i, also for n sufficiently large $u^i(\hat{\sigma}_i, \sigma_{-i}^n) > u^i(b_i^n, \sigma_{-i}^n)$. This contradicts $b_i^n \in B_i(\sigma_{-i}^n)$.

“a sequence of best-responses converges to a best-response”

Best Response Correspondence Example

	L ($\sigma_2(L) = q$)	R
U ($\sigma_1(U) = p$)	1,1	0,0
D	0,0	1,1



Mixed Strategies: The Kitty Genovese Problem

Description of the problem

Model:

n people all identical

benefit if someone calls the police is x

cost of calling the police is 1

Assumption: $x > 1$

Look for symmetric mixed strategy equilibrium where *p* is probability of each person calling the police

p is the symmetric equilibrium probability for each player to call the police

each player *i* must be indifferent between calling the police or not

if *i* calls the police, gets $x-1$ for sure.

If *i* doesn't, gets 0 with probability $(1-p)^{n-1}$, gets x with probability $1-(1-p)^{n-1}$

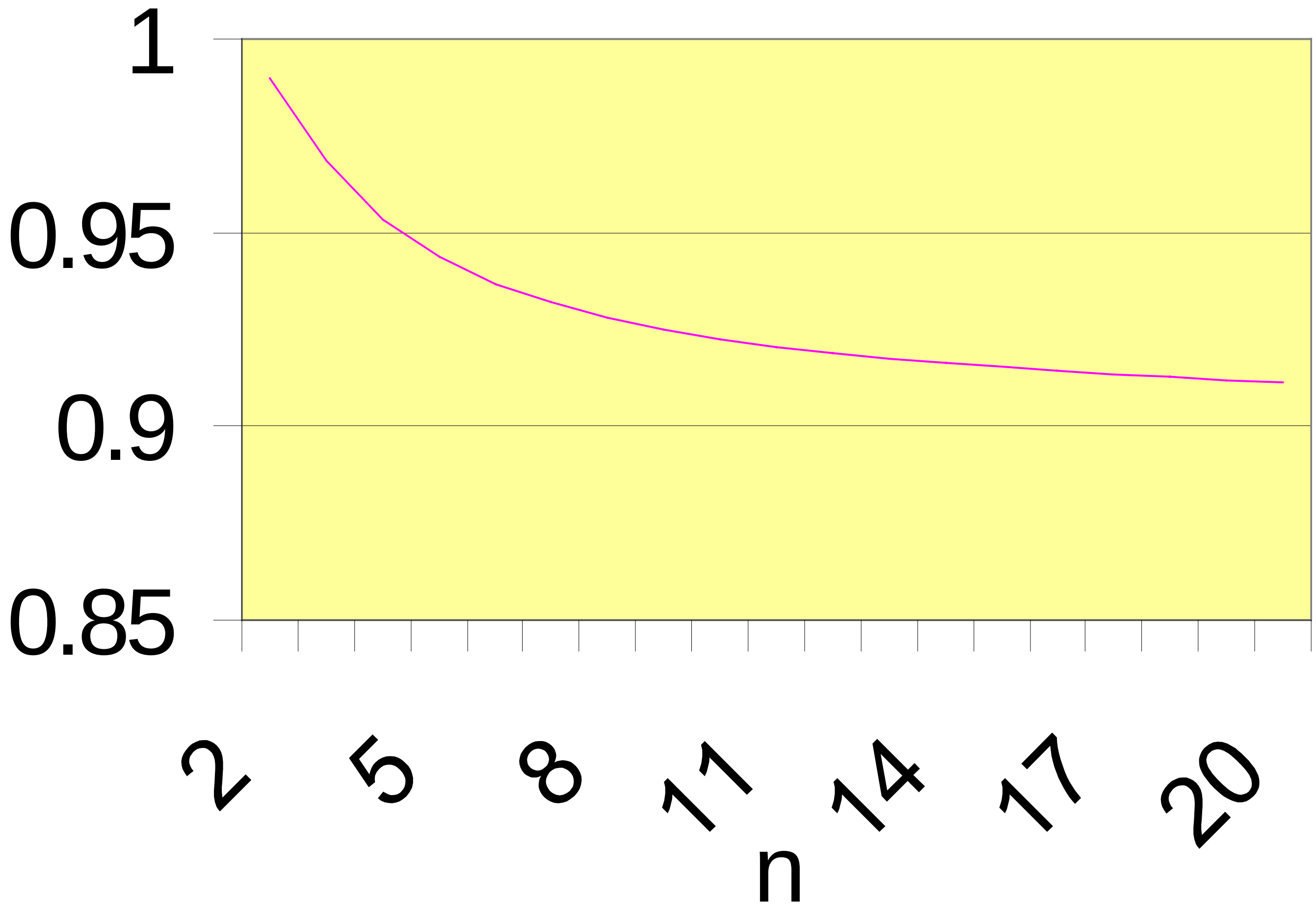
so indifference when $x-1=x\big(1-(1-p)^{n-1}\big)$

solve for $p=1-(1/x)^{1/(n-1)}$

probability police is called

$$1-(1-p)^n = 1-\left(\frac{1}{x}\right)^{\frac{n}{n-1}}$$

probability police are called



x=10

Coordination Games

	L	R
U	1,1	0,0
D	0,0	1,1

three equilibria (U,L) (D,R) (.5U,.5R)

too many equilibria?? introspection possible?

the rush hour traffic game – introspection clearly impossible, yet we seem to observe Nash equilibrium

equilibrium through learning?

Coordination Experiments

Van Huyck, Battalio and Beil [1990]

Actions $A = \{1, 2, \ldots \overline{e}\}$

Utility $u(a_i, a_{-i}) = b_0 \min(a_j) - b a_i$ where $b_0 > b > 0$

Everyone doing a' the same thing is always a Nash equilibrium

$a' = \overline{e}$ is efficient

the bigger is a' the more efficient, but the "riskier"

a model of "riskier" some probability of one player playing $a' = 1$

story of the stag-hunt game

$\overline{e} = 7$, 14-16 players

treatments: A $b_0 = 2b$

B $b = 0$

In final period treatment A:

77 subjects playing $a_i = 1$

30 subjects playing something else

minimum was always 1

In final period treatment B:

87 subjects playing $a_i = 7$

0 playing something else

with two players $a_i = 7$ was more common

1/2 Dominance

Coordination Game

	L (p_2)	R
U (p_1)	2,2	-10,0
D	0,-10	1,1

risk dominance:

indifference between U,D

$$2p_2 - 10(1-p_2) = (1-p_2)$$
$$13p_2 = 11, p_2 = 11/13$$

if U,R opponent must play equilibrium w/ 11/13

if D,L opponent must play equilibrium w/ 2/13

½ dominance: if each player puts weight of at least ½ on equilibrium strategy, then it is optimal for everyone to keep playing equilibrium

(same as risk dominance in 2x2 games)

Trembling Hand Perfection

σ is trembling hand perfect if there is a sequence $\sigma^n >> 0, \sigma^n \to \sigma$ such that

if $\sigma^i(s^i) > 0$ then s^i is a best response to σ^n

Note: thp is necessarily a Nash equilibrium

Examples:

strict Nash equilibrium is always thp

completely mixed Nash equilibrium is always thp

Correlated Equilibrium

Chicken

6,6	2,7
7,2	0,0

three Nash equilibria (2,7), (7,2) and mixed equilibrium w/ probabilities (2/3,1/3) and payoffs

(4 2/3, 4 2/3)

6,6	2,7
7,2	0,0

correlated strategy

1/3	1/3
1/3	0

is a correlated equilibrium giving utility (5,5)

What is public randomization?

Approximate Equilibria and Near Equilibria

- exact: $u_i(s_i | \sigma_{-i}) \geq u_i(s_i' | \sigma_{-i})$

  approximate: $u_i(s_i | \sigma_{-i}) + \varepsilon \geq u_i(s_i' | \sigma_{-i})$

- Approximate equilibrium can be very different from exact equilibrium

Radner's work on finite repeated PD

gang of four on reputation

upper and lower hemi-continuity

A small portion of the population playing "non-optimally" may significantly change the incentives for other players causing a large shift in equilibrium behavior.

Quantal Response Equilibrium
(McKelvey and Palfrey)

propensity to play a strategy

$$p_i(s_i) = \exp(\lambda_i u_i(s_i, \sigma_{-i}))$$

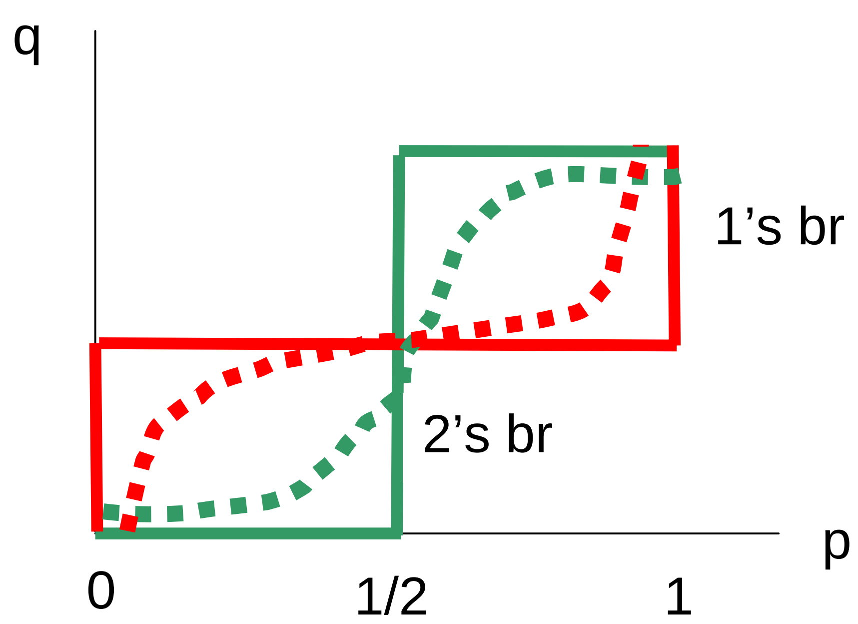
$$\sigma_i(s_i) = p_i(s_i) / \sum_{s_i'} p_i(s_i')$$

as $\lambda_i \to \infty$ approaches best response

as $\lambda_i \to 0$ approaches uniform distribution

Smoothed Best Response Correspondence Example

	L ($\sigma_2(L) = q$)	R
U ($\sigma_1(U) = p$)	1,1	0,0
D	0,0	1,1



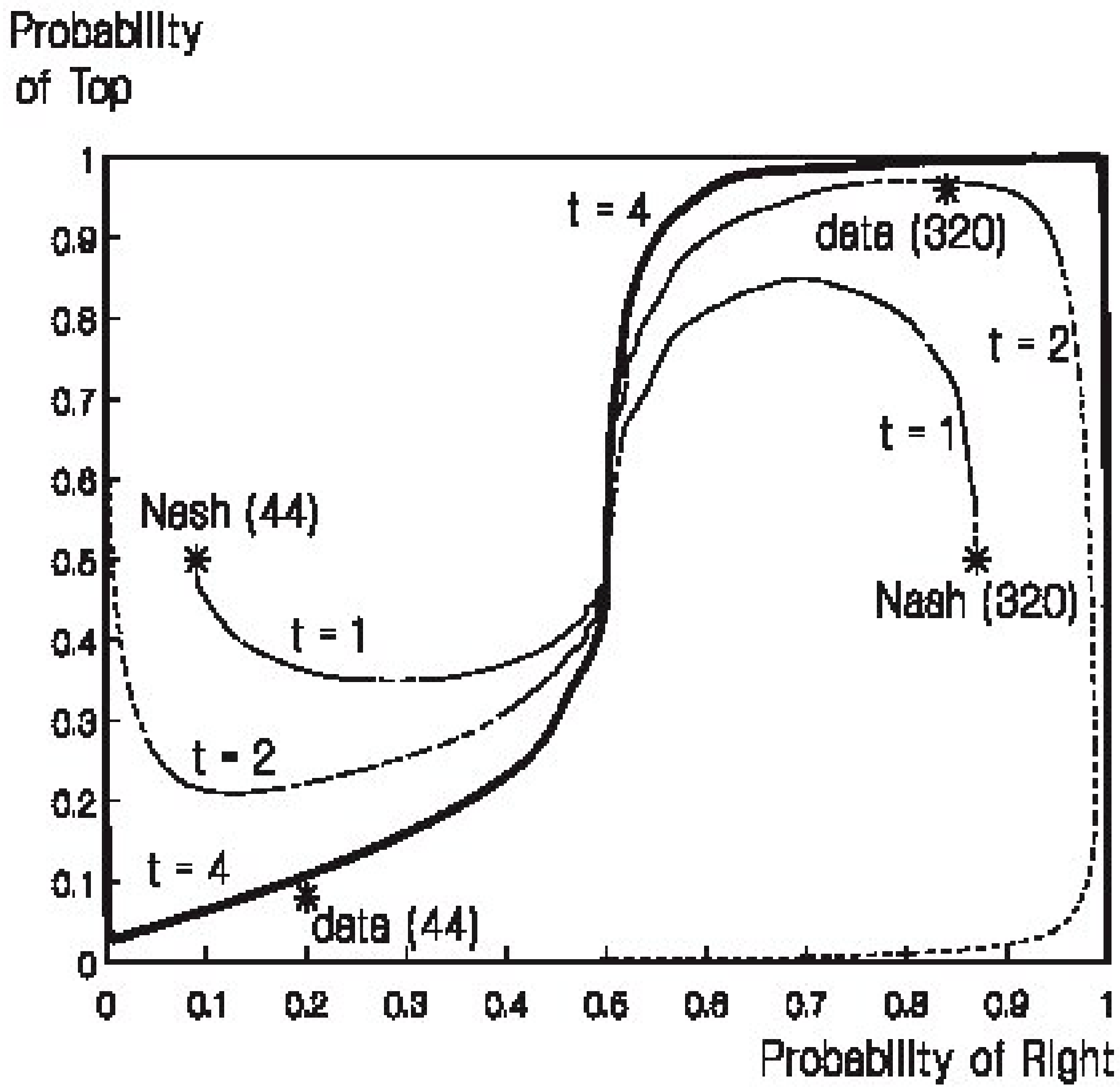
Goeree and Holt: Matching Pennies

Symmetric

	50% (48%)	50% (52%)
50% (48%)	80,40	40,80
50% (52%)	40,80	80,40

	12.5% (16%)	87.5% (84%)
50% (96%)	320,40	40,80
50% (4%)	40,80	80,40

	(80%)	(20%)
50% (8%)	44,40	40,80
50% (92%)	40,80	80,40


Probability
of Top
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Probability of Right
t = 4
data (320)
t = 2
t = 1
Nash (44)
Nash (320)
t = 1
t = 2
t = 4
data (44)