Pay-As-Bid Auctions in Theory and Practice

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Thesis submitted for assessment with a view to obtaining the degree of
Doctor of Economics of the European University Institute

Florence, 26 July 2018

European University Institute
Department of Economics

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Abstract

The pay-as-bid auction, also called the discriminatory price auction, is among the most common auction formats to price and allocate assets and commodities. Trillions of dollars each year are traded in pay-as-bid auctions. The format is the natural multi-unit extension of the first-price auction of a single item. Bidders specify a price for each unit they want to buy. The market clears at the price where supply intersects aggregate demand and winning bidders pay their bids for each unit won. In the first chapter of my thesis, I explain strategic differences and similarities between the single-item and multi-unit case. In practice, it is rare that multi-unit auctions take place in isolation. The second chapter introduces a model of interconnected pay-as-bid auctions. The auctions run in parallel and offer perfectly divisible substitute goods to the same set of symmetrically informed bidders with multi-unit demand. This connects the demand side of both auctions. The supply side is linked because the total amounts for sale may be correlated. I show that there exists a unique symmetric Bayesian Nash equilibrium when the marginal distributions of supply have weakly decreasing hazard rates. I then develop practical policy recommendations on how to exploit the interconnection across auctions to increase revenues. These theoretic insights are the basis for the final chapter of my thesis. In collaboration with Jason Allen (Bank of Canada) and Jakub Kastl (Princeton University) I use data from auctions of Canadian debt to quantify the extent to which demands for securities with different maturities are interdependent. Generalizing methods for estimating demand schedules from bidding data to allow for interdependencies, our results suggest that 3, 6 and 12-month bills are often complementary in the primary market for Treasury bills. We present a model that captures the interplay between the primary and secondary markets to provide a rationale for our findings.

Acknowledgement

I THANK David Levine for guiding me towards better research, struggling through messy proofs and keeping an eye on what is essential; Peter Cramton to whom I owe my fascination for multi-unit auctions; Robert Wilson whose creativity and ability to break down complex arguments into much simpler pieces continues to surprise me; Paul Milgrom for broadening my intellectual horizon; as well as the numerous other researchers, family and friends who supported me and my work throughout the past four years.

Thank you!

Für Monika
Ohne Dich wäre ich nicht (die Gleiche).

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1.1 Introduction

The pay-as-bid auction, also known as discriminatory price auction, is a popular mechanism for allocating assets and commodities worldwide. It extends the rules of the well-known first-price auction to the sale of multiple units of the same good: Bidders submit bidding schedules which specify a price for each unit they demand. Individual demands are then aggregated by the auctioneer to determine the market clearing price above which all bids win. All winners pay "as-they-bid" for all units they won. The pay-as-bid auction is very popular among governments and central banks. It is used to allocate Treasury bonds and implement other operations such as Quantitative Easing on the open market. Outside the financial sector it distributes carbon credits as well as electricity generation in several countries.[1] In total, trillions of dollars are transferred every year using this type of auction. Despite its importance, we know little about strategies used by auction participants. Expect in special circumstances we are even unable to compute best-response strategies (Woodward (2015)). To a large extent the literature focuses on the case of single-unit demand. Assuming that each bidder wants at most one unit is a simplifying assumption that is violated in most real-world applications. A bank bidding in a Treasury auction, for instance, clearly wants more than just a single dollar worth of the offered Treasury bill. With multi-unit demand bidding strategies in pay-as-bid auctions are, according to a common understanding of the recent literature, more complicated than those in its single-unit counterpart, the first-price auction.[2] The reason is that changing one's n'th bid may affect not only whether the n'th unit is won, but also the bidder's belief of where the market will clear. To optimize their payoffs, bidders have incentives to shade their bids for each unit differently. This behavior, known as "strategic bid shading" or "demand reduction", is by design not present in auctions of a single, indivisible object. It is seen as "the key to why the analogy between single-unit and multi-unit auctions does not apply"(Ausubel et al. (2014), p. 1367).

[1] For more details see Brenner et al. (2009), Bartolini and Cottarelli (1997), Ghazizadeh et al. (2007), Maurer and Barroso (2011).

[2] "Except in the case where bidders have demand for only a single unit of the auctioned commodity, the analysis of multi-unit auctions are [...] more difficult than that of single-unit auctions (Hortaçsu (2011), p. 345)."

In a simple theoretic framework in which bidders with multi-unit demand compete for shares of a perfectly divisible good, I argue that the complexity of pay-as-bid auctions comes not from demand reduction but more specifically type-dependent demand reduction. In my benchmark model $N \geq 2$ risk-neutral bidders are symmetrically informed. They all share the same type, known to them but not the seller, but are uncertain about the total amount of the good that will be for sale.[3] I discover the following analogy between pay-as-bid and first-price auctions: Each of N symmetrically informed bidders shades his bid in the symmetric equilibrium of the pay-as-bid auction for 1 of N shares of the perfectly divisible good *as if* he competed with $(N-1)N$ bidders for one indivisible good in a canonical first-price auction with independent private types. This analogy might break when bidders are not symmetrically informed but have private information, i.e. types. Whether bidding in pay-as-bid auctions is more complex thus depends on the source of uncertainty bidders face. With private information it can be optimal for bidders of different types to reduce demand in different ways. This suggests that it is not demand reduction (or differential bid shading) per se that makes bidding choices in pay-as-bid auctions more difficult. What gives rise to complicated equilibrium effects seems to be type-dependent demand reduction instead. Such type dependency introduces asymmetric trade-offs not only across units of the good, but also agents. It therewith generates complications that have no equivalent in single-unit auctions.

My findings build on an intuitive bid-representation theorem for pay-as-bid auctions. It characterizes the functional form of the bidding schedule when bidders are symmetrically informed (benchmark model) and - with some limitations - when they have private information (model extension). In future work my theorem might serve as basis to construct equilibrium strategies for other, potentially more general environments, with asymmetrically informed bidders that have multi-unit demand. Computing such equilibrium strategies for pay-as-bid auctions is still an open question in the literature (Hortaçsu and Kastl (2012)). To illustrate how to use my bid-representation theorem to construct equilibria, I conclude the article by deriving an equilibrium in linear bidding strategies. This equilibrium is new to the literature. It is the counterpart to Ausubel et al. (2014)'s linear equilibrium without private information and helps us understand which role private information plays for strategic incentives in pay-as-bid auctions: In the symmetric equilibrium, privately informed agents bid like symmetrically informed agents who all draw the the lowest type, but add a type-specific discount factor.

More generally, my work could be a first step into establishing a more general theoretic connection between bidding in first-price and bidding in pay-as-bid auctions with multi-

[3] This modeling assumption nicely reflects two common features of real-world pay-as-bid auctions. For one, the amount to be allotted is in some cases, such as Treasury auctions in Germany, Greece, Belgium, Turkey or Sweden (Brenner et al. (2009)), adaptable during the auction. Secondly, the total supply is often shaped by so called "non-competitive" tenders. These are irregular bids in that only a quantity is specified. The price is determined automatically. It is either the average price paid by (regular) bidders or the market clearing price. How many non-competitive tenders will be served is unknown to the (regular) bidders so that the total amount for sale that matters for them is random.

unit demand. Such a methodological link would increase our poor knowledge of pay-as-bid auctions. We could re-visit the literature on first-price auctions. In contrast to pay-as-bid auctions, first-price auctions have been at the core of auction theory since the very beginning (Vickery (1961)). They have been studied extensively and are well understood. When bidders have independent private values, we know, for instance, that first-price auctions can be revenue equivalent to second-price auctions, and that they are strategically equivalent to the Dutch auction. For the pay-as-bid auction we know much less. We do not know whether, and if so under which conditions, it might be strategically or revenue equivalent to another auction format, for instance the uniform-price auction. It differs from the pay-as-bid auction only in that bidders pay the market clearing price for all units they win, instead of their individual bids. The existing literature has not come to a consensus on which of the two auctions are more efficient or bring higher revenue.[4] By finding conditions that influence the distribution of winning quantities (which is at the center of my analysis) it might be possible to distinguish cases under which either auction format dominates the other and explain why.

Related Literature. Building on the literature of "share auctions", put forward by Wilson (1979) and further developed most notably by Back and Zender (1993) and Wang and Zender (2002),[5] my analysis of the benchmark model with symmetrically informed bidders is complementary to Pycia and Woodward (2017). In independent work, we derived the functional form of the equilibrium bidding function under the simplifying assumption that bidders are symmetrically informed. Relative to previous studies, such as Wang and Zender (2002) as well as Ausubel et al. (2014), our result is more general in that we neither impose marginal utility to be linear nor total supply to be distributed according to the Pareto distribution. Instead, our theorem holds under a very broad class of utility functions and distributions. It is similar to Holmberg (2009) who studies pay-as-bid procurement auctions with general cost functions (here utility functions) and perfectly inelastic demand (here supply). He shows that an equilibrium exists if the hazard rate of demand is monotonically decreasing and bidders have non-decreasing marginal costs. Pycia and Woodward (2017) go one step further in bringing attention to more general sufficient conditions under which equilibrium existence is guaranteed. As such, their work has been acknowledged as the best unique existence result currently available for pay-as-bid auctions (Hortaçsu and McAdams (2018)). Before focusing on the design of pay-as-bid auctions by optimally choosing the distribution of total supply and a reserve price, Pycia and Woodward (2017) show that the equilibrium is symmetric, strictly monotone and differentiable in quantity rather than assuming those properties. Contrary to my work, they do not explain the shape of the bidding function in any detail. This could come from a difference in the way we express the bidding function. Their representation highlights that a "bid for any quantity is a weighted average of the

[4]See Ausubel et al. (2014), Pycia and Woodward (2017) for theoretic contributions and Février et al. (2004), Kang and Puller (2008), Armantier and Sbaï (2006, 2009), Hortaçsu and McAdams (2010) for empirical comparisons.

[5] Earlier contributions with indivisible goods include Katzmann (1995), Engelbrecht-Wiggans (1998), Swinkels (2001), Engelbrecht-Wiggans and Kahn (2002), Lebrun and Tremblay (2003), Chakraborty (2004, 2006), Anwar (2007).

bidder's marginal values for this and larger quantities, where the weights are independent of the bidder's marginal values" (p. 4). Mine, instead, underlines the direct connection of bidding behavior to the first-price auction. Therewith I am, to the best of my knowledge, the first to discover this particular linkage. As my main goal is to work out this correspondence rather than to solve the most general model of pay-as-bid auctions, I will make more stringent technical assumptions that simplify the mathematical derivations than I would need to. I invite who is interested in a more general framework with symmetrically informed bidders to consult Pycia and Woodward (2017).

In contrast to Holmberg (2009) and Pycia and Woodward (2017) I make first steps towards an auction environment in which bidders are asymmetrically informed. In a model extension, each bidder draws an independent private type. While the benchmark model with identical bidders is helpful to fix ideas and identify key factors that drive bidding behavior in the multi-unit auction, it is not so useful for evaluating performance. Auctions are typically run to extract individual information from agents, so as to allocate resources to those who benefit the most at the highest price possible. Only a framework with private information allows us to analyze information aggregation and efficiency. Furthermore, the extension towards a framework with independent private values helps to close the gap between theoretic and related empirical work which tends to build on models with private values. Starting with Hortaçsu (2002) researchers have estimated the bidders' private, marginal willingness to pay in multi-unit auctions (see also Février et al. (2004), Hortaçsu and McAdams (2010), Kastl (2011, 2012), Hortaçsu and Kastl (2012), Cassola et al. (2012), Hortaçsu et al. (2018)). The structural estimation approach is based on an implicit characterization of the bidding function in form of the first-order conditions. For each unit-bid, these necessary conditions have been recognized to capture a similar trade-off to the one in a first-price auction, where bidders trade-off the probability of winning against their gain from it (Kastl (2017)). Unfortunately, these first-order conditions are not informative for a theorist. The reason is that they depend on the distribution of the market clearing price. When bidders have private types it can be defined only implicitly (for any given set of strategies) via market clearing. The econometrician is able to simulate this distribution from the data. The theorist is not. Woodward (2016) nicely reflects the state of the art on pay-as-bid share auctions with private types. He proves equilibrium existence in pay-as-bid auctions with private types without specifying the equilibrium bidding function. He shows that bidders might have incentives to "iron", that is flatten, their bidding functions for small amounts which they are certain to win.

In the remainder of the article, Section 1.2 sets up the benchmark model with symmetrically informed bidders and states the bid-representation theorem. It builds the basis for the comparison of bidding in the pay-as-bid auction to bidding in the canonical first-price auction (Section 1.3). I then provide an extension of the main result to an environment with independent private values (Section 1.4). Before concluding in Section 1.5, Section 1.4.1 focuses on a linear example. All proofs are given in Appendix A. Random variables will be highlighted in **bold** throughout the article.

1.2 Benchmark Model

$N \geq 2$ risk-neutral bidders participate in a pay-as-bid auction. They share the same type $\boldsymbol{t}$ drawn from some commonly known distribution. It is unknown to the seller. From the perspective of the bidder this common type has no strategic relevance because it is known to all of them. It is fixed at some value t throughout the analysis. Instead, bidders are uncertain about the total amount of the perfectly divisible good that is for sale, $\boldsymbol{Q}$. Independent of the bidders' type, it is drawn from some commonly known, non-degenerated and twice-differentiable distribution $F_Q(\cdot)$ with bounded support $[0, \overline{Q} > 0]$ and strictly positive density $f_Q(\cdot)$.[6] Imposing a zero lower bound will simplify the analysis later on. It will rule out that bidders have incentives to iron their bids when they have private types (see Woodward (2016)). In practice the zero lower bound could come from a non-zero probability that the auction is cancelled.

Consuming quantity q generates utility for each bidder. The marginal utility $v(q)$ represents the bidder's true marginal willingness to pay for this amount. $v(\cdot)$ is strictly decreasing, and twice differentiable. Agents can have a satiation quantity, q^s. This is the amount at which the agent's marginal valuation turns 0: $v(q) = 0$ for $q \geq q^s$. It is assumed to be large, $q^s \geq \overline{Q}/N$, for simplicity. If $q^s \to \infty$, winning some more at a price of zero is always better.

Based on his true marginal willingness to pay each bidder submits a weakly decreasing and differentiable bidding function: $b_i(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$. It is an inverse demand, mapping from the quantity-space into the space of prices. The corresponding demand function maps from prices to quantities. It is denoted by $x_i(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$.

Once all bidders have submitted their individual demands, each market clears at the minimal price for which the aggregate demand of all bidders meets the realized total supply Q. If the aggregate demand exactly equals the total supply at the market clearing price p^c, each bidder i wins the quantity he demanded at this clearing price: $Q = \sum_i x_i(p^c)$ with $p^c = b_i(q_i^c)$. In that case, all winners pay what they were willing to pay for all units won, abbreviated by $q_i^c \equiv x_i(p^c)$: $\int_0^{q_i^c} b_i(x)dx$. Otherwise, if the aggregate demand at the clearing price is higher than the total supply, bidders have to be rationed according to some tie-breaking rule. In equilibrium no one will have to be rationed because bidding function will be strictly decreasing. This ensures that the market always clears exactly. Which tie breaking rules is used is therefore irrelevant.

From an ex-ante perspective agents do not know how much they will win nor at which price

[6] Relying on an idea by Pycia and Woodward (2017) Appendix A.4 shows that results extend to distributions with unbounded support in presence of an arbitrarily small but positive reserve price. Since the agents' true marginal willingness to pay is decreasing by assumption, it will drop below the reserve price at some point. The support of the quantity that will matter for bidding decisions is therefore bounded endogenously. Without positive reserve price and unbounded support, the bidder's objective functional might not be well-defined because the expectation of the bidder's total surplus might not exist.

the market will clear. Both the clearing price $\boldsymbol{p^c}$ and the clearing price quantity $\boldsymbol{q_i^c}$ depend on how much there will be for sale $\boldsymbol{Q}$. This amount is random. The adequate solution concept is therefore Bayesian Nash Equilibria. I focus on equilibria in pure-strategies. They consist of a set of bidding functions $\{b_i^*(\cdot)\}_{i=1}^N$ that maximize each bidder i's expected total surplus from winning the ex-ante unknown clearing price quantity $\boldsymbol{q_i^c}$ given all others $j \neq i$ choose $b_j^*(\cdot)$. This total surplus is the difference between the bidder's total utility from winning the clearing price quantities $\int_0^{\boldsymbol{q_i^c}} v(x)dx$ and his total payments $\int_0^{\boldsymbol{q_i^c}} b_i(x)dx$

Definition 1. *A pure-strategy Bayesian Nash Equilibrium (BNE) is a set of bidding functions such that* $b_i^*(\cdot) \in \arg\max_{b_i(\cdot)} \mathbb{E}\left[\int_0^{\boldsymbol{q_i^c}} v(x) - b_i(x)dx\right] \;\; \forall i \in N.$

Given the symmetric environment it is natural to restrict attention to symmetric equilibria. In such equilibria bidders share the total supply equally. Later on, it will be convenient to work with the agent's "equilibrium winning quantity", instead of the total supply:

$$\boldsymbol{q^*} \equiv \frac{\boldsymbol{Q}}{N} \in \left[0, \frac{\overline{Q}}{N}\right] \equiv [0, \overline{q}^*].$$

Its marginal distribution and density will be denoted by $F_{q^*}(\cdot)$ and $f_{q^*}(\cdot)$.

Having introduced the environment, I turn to the core of the article. I first derive my main result for the benchmark model before generalizing it to an environment with private information.

1.3 Pay-As-Bid vs. First-Price Auctions

My goal is to highlight differences and similarities in bidding strategies between pay-as-bid and first-price auctions. The following bid-representation theorem will serve as basis for the discussion. The Appendix shows that the bidding function is equivalent to Pycia and Woodward (2017)'s Theorem 3.

Theorem 1. *Consider distributions of total supply with weakly decreasing hazard rate. There exists a pure-strategy Bayesian Nash equilibrium in which all bidders submit*

$$b^*(q) = v(q) - \left(\int_q^{\overline{q}^*} \left[\frac{1 - F_{q^*}(x)}{1 - F_{q^*}(q)}\right]^{\frac{N-1}{N}} (-1) \left(\frac{\partial v(x)}{\partial q}\right) dx\right) \quad \textit{on } [0, \overline{q}^*] \tag{1.1}$$

and $b^*(q) = v(q)$ *for* $q \in (\overline{q}^*, \infty)$.

The equilibrium exists if total supply is drawn from a distribution with weakly decreasing hazard rate which implies that the distribution of winning quantities F_{q^*} has this property. This existence condition is known in the literature (see Holmberg (2009)). It ensures that bidders do not have incentives to deviate from the equilibrium strategy. Technically it is a sufficient condition in the maximization problem that each bidder solves to determine his best reply. Recently, Pycia and Woodward (2017) have derived a weaker condition. They also show that (1.1) is the only function that can arise in any (not necessarily symmetric) equilibrium on the domain of relevant quantities $q \in [0, \overline{q}^*]$. Higher amounts are unachievable. Since no agent ever wins these high amounts nor pays for them, they are out of equilibrium. The bidder's choice for those high amounts is irrelevant as long as his bidding function is decreasing on the whole domain $\mathbb{R}^+$. Here I consider the most natural equilibrium in which the agents bid truthfully for unfeasibly large quantities.[7]

For attainable quantities, the bidding function (1.1) is surprisingly simple. Because the bidder "pay-as-he-bids" he understates his true marginal willingness to pay for each unit that he might purchase in equilibrium: $v(q)$. This is similar to an independent private-value sealed-bid first-price auction, where bidders shade their true types.

The symmetric equilibrium of a canonical first-price auction with $N \geq 2$ bidders, each drawing an independent private value $s \in [0, \overline{S}]$ from a common distribution $F_s(s)$, is well known. Given his true marginal willingness to pay for the indivisible object, $v(s) = s$, the bidder submits

$$\beta^*(s) = v(s) - \left(\int_0^s \left[\frac{F_s(x)}{F_s(s)} \right]^{N-1} \left(\frac{\partial v(x)}{\partial s} \right) dx \right) \quad on \quad \left[0, \overline{S}\right]. \tag{1.1b}$$

The strategy function maps the agent's true type into his price offer. Whoever offers the highest price wins the object.

Comparing the bidding functions (1.1) with (1.1b) reveals the close similarity between bidding behavior in the pay-as-bid and first-price auction. To see it, however, one must eliminate two differences that distinguishe the two functions due to differences in the two set-ups. First, the uncertainty that bidders face comes from different sources. In the first-price auctions agents have private types. The bidder wins if he has the highest private value: $s \geq \boldsymbol{s_j}\ \forall j \neq i$. In the stylized pay-as-bid auction there are no private types. The equilibrium quantities, each representing a share of the perfectly divisible good, take their place. A bidder now wins q when the market has not cleared yet: $Nq \leq \boldsymbol{Q} = N\boldsymbol{q^*}$. To draw the analogy between both auction formats one must compare the type $\boldsymbol{s}$ with the equilibrium share $\boldsymbol{q^*}$ and the corresponding probabilities that determine whether the agent wins or not:

$$\boldsymbol{s} \leftrightarrow \boldsymbol{q^*} \text{ and } F_s(\cdot) \leftrightarrow 1 - F_{q^*}(\cdot). \tag{1.2}$$

Second, the agent's true valuation for the object is strictly *increasing* in the first-price auction, while it is strictly *decreasing* in the pay-as-bid auction. This inverts the bounds of

[7]One other alternative is to submit a flat function at value $v(\overline{q}^*)$ for unattainable quantities. This equilibrium, however, is not robust to bidders trembling. In case some agent makes a mistake which leads some other other to win such large amounts, the later would make a loss winning as he would pay a higher price than he truly values these additional units.

the integrals. In the pay-as-bid auction the integrals go from the realization q to its largest possible value; in the first-price auction, from the smallest value 0 of the realization to the draw s:

$$\int_0^s ...v'(x)dx \leftrightarrow \int_q^{\overline{q}^*} ...(-1)v'(x)dx. \tag{1.3}$$

Bearing (1.2) and (1.3) in mind, the two bidding functions differ in one element only: The exponent of the bidding function of the pay-as-bid auction is $\frac{N-1}{N}$, the one of the first-price auction is $N-1$. In case $N-1$ would equal $(N-1)N$ the bidding function of the first-price auction would be analogous to the one in the pay-as-bid auction. This gives rise to the following observation.

Main Result 1. *In the symmetric equilibrium of the pay-as-bid auction with symmetrically informed bidders, each bidder shades his bid for 1 of N shares as if he competed with $(N-1)N$ bidders for 1 indivisible good in a first-price auction with independent private values.*

The result is intuitive: In a single good first-price auction, uncertainty over types can be aggregated. A bidder effectively bids against one other bidder whose type is a random variable with the same distribution as the highest order statistic of the common distribution of types. In this sense he chooses his bid given the residual demand curve. Crucially for a first-price auction, he bids as if he wins the auction because he pays his bid if and only if he wins. What about a multi-unit pay-as-bid auction? Here he also takes the residual demand of all others as given. In a symmetric equilibrium, he is guaranteed to win $\frac{\boldsymbol{Q}}{N}$, so he always bids as if he wins $\frac{\boldsymbol{Q}}{N}$. Now, $\boldsymbol{Q}$ is uncertain, so we have to think of it slightly differently. More precisely, it is optimal to bid as if he wins the marginal share. In this regard the bidder is playing like in a single item first-price auction "on the margin". Low supply looks like a high type aggregate opponent; the math doesn't distinguish where this uncertainty comes from. A natural question to ask is whether this result holds when bidders have private information in both auction formats.

1.4 Extension: Private Information

I make the following adjustments in the set-up: Each bidder now draws an independent private type from the same, commonly known distribution with twice differentiable distribution on bounded support $[\underline{t} > 0, \overline{t}]$ and strictly positive density. Bidder type t_i derives a true marginal value $v(q, t_i)$ from amount q. As above, it is strictly decreasing and twice differentiable in quantity, plus integrable in the type. If it hits the zero line at some finite satiation quantity it remains zero. Having observed their type, all agents submit a type-dependent bidding function $b_i(\cdot, t_i) : \mathbb{R}^+ \to \mathbb{R}^+$. It is weakly decreasing and differentiable in quantity. With these adaptations, a BNE in pure strategies is defined analogously to Definition 1.

Theorem 2. *In a symmetric pure-strategy Bayesian Nash equilibrium bidders submit*

$$b^*(q,t_i) = v(q,t_i) - \int_q^{\overline{q}_i^*} \left[\frac{1-F_{q_i^*}(x)}{1-F_{q_i^*}(q)}\right]^{\frac{N-1}{N}} (-1) \left(\frac{\partial v(q,t_i)}{\partial q}\right) dx \text{ on } q \in [0, \overline{q}_i^*] \tag{1.4}$$

and $b^*(q,t_i) = v(q,t_i)$ *on* $q \in (\overline{q}_i^*, \infty)$. *This equilibrium exists if*

(i) distributions of total supply and types are such that the amount an agent wins in the symmetric equilibrium $\boldsymbol{q_i^*}$ *is drawn from a distribution* $F_{q_i^*}$ *with weakly decreasing hazard rate and strictly positive density on support* $[0, \overline{q}_i^*]$ *and*

(ii) the corresponding demand schedule $x^*(\cdot, t_i) = b^{-1*}(\cdot, t_i)$ *is additively separable in* t_i.

In the symmetric equilibrium with private information, agents no longer split the total supply equally, $\boldsymbol{q^*} \equiv \frac{\boldsymbol{Q}}{N}$. The amount an agent wins now depends on his type: $\boldsymbol{q^*}(t_i)$ abbreviated by $\boldsymbol{q_i^*}$. The equilibrium bidding function (1.4) has the same shape as function (1.1) without private information. Whether this equilibrium exists depends on the underlying distributions of total supply and types as well as the number of participating bidders. Both determine the distribution of i's winning quantity $F_{q_i^*}(\cdot)$. Its shape in turn will determine whether the bidding function of each type (1.4) assumes an inverse function (the demand function) that is additively separable in the type. Without private types, the existence conditions boil down to the assumption that total supply (and with it the winning equilibrium quantity) is drawn from a distribution with decreasing hazard rate (as in Theorem 1). Determining general conditions on the primitives of the model that guarantee existence of this equilibrium is beyond the scope of this article. The generalized theorem, instead, is meant to underline differences and similarities between pay-as-bid auctions relative to first-price auctions in presence of private information. In line with the previous section it allows me to make the following observation.

Main Result 2. *In the symmetric equilibrium of the pay-as-bid auction with independent private types, each bidder shades his bid for 1 of* N *shares as if he competed with* $(N-1)N$ *bidders in a first-price auction with independent private values provided the submitted demand function is additively separable in their type and strictly decreasing in price.*

My analysis highlights a complication in multi-unit auctions that has, to the best of my knowledge, not yet been made explicit in the literature. Strategizing in pay-as-bid auctions might not be as "simple" as bidding in first-price auctions when agents of different types submit demands with different slopes.[8] Intuitively, a type-dependent slope introduces an asymmetry in incentives not only across prices but also agents with different types. Now type t_i reduces his true demand at price p by a different amount than type t_j. In other words, bidders do not only reduce their demand differently across prices but each type does

[8]Even though I do not show that this necessary condition is also sufficient, I expect that it is not possible to derive an equilibrium bidding function in the pay-as-bid auction that has the discussed similarities to the one of the first-price auction.

it differently. It seems to be the type-dependency that creates complicated equilibrium effects, not demand reduction per se.

As in Pycia and Woodward (2017)'s model without private information, the theorem and with it the main result extend to auctions with reserve prices $R > 0$, where total supply or types may be drawn from distributions with potentially unbounded supports (see Appendix A.4). This insight could be valuable for the optimal design of pay-as-bid auctions. For first-price auctions where bidders draw independent private types $s \in [0, \overline{S}]$ from a common distribution $F_s(s)$ the formula for the optimal reserve-price is well known: $R - \left(\frac{1-F_S(R)}{f_S(R)}\right) = 0$. For pay-as-bid auctions, we do not know how to set reserve prices optimally. My findings might help to determine an analogous formula for an optimal reserve price in pay-as-bid share auctions in presence of private information.

1.4.1 Linear Example

To conclude I illustrate how Theorem 2 can be used to find equilibria of pay-as-bid share auctions in presence of private information. It is, to the best of my knowledge, new to the literature. My approach of finding it could be used in other set-ups.

My aim is to construct a linear example. In search for an equilibrium with a linear bidding function, it is natural to assume that the agents' true marginal willingness to pay is linear: $v(q, t_i) = \max\{t_i - \rho q, 0\}$ with $\rho > 0$. Assuming linear marginal values alone, however, is not enough to generate linear bidding strategies. To see this, recall the bid-representation of Theorem 2. From there we know that the agent's function depends nontrivially on the distribution of his winning quantity $F_{q_i^*}(\cdot)$. With linear marginal valuations function (1.4) becomes

$$b^*(q, t_i) = t_i - \rho q - \rho \int_q^{\overline{q}_i^*} \left[\frac{1 - F_{q_i^*}(x)}{1 - F_{q_i^*}(q)}\right]^{\frac{N-1}{N}} dx. \tag{1.4}$$

For many distributions the integral, and with it the bidding function, will not be linear in quantity. For an auction environment without private types, Ausubel et al. (2014) show that equilibria are linear only if the per-capita supply (here referred to as i's equilibrium winning quantity, $\boldsymbol{q^*}$) is drawn from the Generalized Pareto Distribution (GDP). This result extends without complications to an environment with private types where $\boldsymbol{q_i^*}$ replaces $\boldsymbol{q^*}$. There are two important differences. First, the distribution of i's winning quantity becomes type dependent. More importantly, it is no longer exogenously given by the distribution of total supply but is endogenous. It is now an equilibrium object itself. In the linear example, where demand schedules take the following form

$$x^*(p, t_i) = a^* + c^* t_i - e^* p \text{ with } a^*, c^* \in \mathbb{R}, e^* > 0 \tag{1.5}$$

it depends, by market clearing, on equilibrium coefficient c^*

$$\boldsymbol{q_i^*} \equiv \frac{1}{N}\left[\boldsymbol{Q} - c^* \sum_{j \neq i} \boldsymbol{t_j} + (N-1)c^* t_i\right]. \tag{1.6}$$

The winning quantity is a transformed convolution of the independent total supply $\boldsymbol{Q}$ and $(N-1)$ iid types, $\boldsymbol{t_j}$, which are weighted by the equilibrium coefficient, $-c^*$. As shown in the following corollary, a linear equilibrium exists when $\boldsymbol{q_i^*}$ follows a Generalize Pareto Distribution. Even though I cannot show that there exist distributions of total supply and types that generate the GDP for i's winning quantity, I am optimistic that there are examples. In particular, I conjecture that it is possible to pick suitable Gamma Distributions for total supply and types. My believe comes from the fact that the GDP belongs to the class of Generalized Gamma Convolution,[9] whose elements can be represented as the distribution of the sum of two or more non-constant (not necessarily identically distributed) random variables which are distributed according to a Gamma Distribution.[10]

Corollary 1. *Let* $v(q, t_i) = \max\{t_i - \rho q, 0\}$ *with* $\rho > 0$*, and asume* $N > \frac{\overline{Q}\rho}{\underline{t}}$.
For distributions of total supply and types under which the amount an agent wins in the symmetric equilibrium $\boldsymbol{q_i^*}$ *is drawn from the Generalized Pareto Distribution*

$$F_{q_i^*}(q) = 1 - \left[\frac{\sigma(\xi, t_i) + \xi q}{\sigma(\xi, t_i)}\right]^{-\frac{1}{\xi}}$$

with scale parameter

$$\sigma(\xi, t_i) = -\xi\left(\frac{N(1-\xi)-1}{N(1-\xi)\rho}\right)(t_i - \underline{t}) - \xi\left(\frac{\overline{Q}}{N}\right)$$

and shape parameter

$$\xi \in (-\infty, -1]$$

there exists a pure-strategy Bayesian Nash equilibrium in which bidders submit

$$b^*(q, t_i) = \begin{cases} \left(\frac{1}{1-\xi}\right)[t_i - \xi\underline{t}] - \left(\frac{\rho}{N(1-\xi)-1}\right)\left[(N-1)q - \xi\overline{Q}\right] & \text{for } q \in [0, \overline{q}_i^*] \\ v(q, t_i) & \text{for } q \in (\overline{q}_i^*, \infty) \text{ with } \overline{q}_i^* \equiv \left(\frac{\sigma(\xi, t_i)}{-\xi}\right). \end{cases}$$

The corollary specifies several restrictions on parameters. Before analyzing how agents bid in equilibrium, I explain why.

Parameter Restrictions. The first restriction, $N \geq \frac{\overline{Q}\rho}{\underline{t}}$, makes sure that the market clears at a non-negative price. In particular, it guarantees that the marginal valuation of the lowest

[9]This class was introduced by Thorin (1977a,b). It is the smallest class of distributions on $\mathbb{R}^+$ that contains Gamma Distributions and is closed with respect to convolution and weak limits. This means that any element of this class is the weak limit of finite convolutions of Gamma Distributions.

Bondesson (1979) showed that distributions that have a density of the form $f(x) = Cx^{\beta-1}(1+cx^{\alpha})^{-\gamma}, x > 0, 0 < \alpha \leq 1$ belong to the class of Generalized Gamma Convolutions. The density of the Generalized Pareto Distribution (with location parameter of 0) is $f(x) = \frac{1}{\sigma}\left(1+\xi\frac{x}{\sigma}\right)^{-\left(\frac{1}{\xi}+1\right)}$, where $\sigma > 0$. It can be written as can be written as $f(x) = \left(\frac{\theta+1}{\delta}\right)\left(1+\frac{\theta x}{\delta}\right)^{-\left(\frac{1}{\theta}+2\right)}$ with $\xi = \frac{\theta}{\theta+1}$ and $\sigma = \frac{\delta}{\theta+1}$. Now, setting $c = \frac{\theta}{\delta}$, $\alpha = \beta = 1$ and $\gamma = \frac{1}{\theta} + 2$ shows that this density takes the form of a of Generalized Gamma Convolution (Hamedani (2013))

[10]In statistical terms, one says that the GCC is self-decomposable.

type is non-negative at the highest quantity he might win in equilibrium: $v(\overline{q}_i^*, \underline{t}) \geq 0$. This in turn ensures that no type will ever submit a bid-price that is negative.[11]

The other two conditions restrict the two shape parameters of distribution $F_{q_i^*}(\cdot)$. For one, ξ must be weakly smaller than -1 to guarantee that no one has incentives to deviate from the equilibrium. This extra condition is needed because the hazard rate of the Generalized Pareto Distribution with bounded support ($\xi < 0$) is *increasing*. So far, I have focused on distributions with *decreasing* hazard rates, where necessary conditions are *always* sufficient. For distributions with increasing hazard rates they may, or may not be. To avoid creating confusion that would have distracted from my main points I have simply imposed the (unnecessarily strict) condition of decreasing hazard rates in the main body of this article.

Secondly, the scale parameter $\sigma(\xi, t_i)$ is not just any positive real number but a function of ξ and t_i. It determines the upper bound of $\boldsymbol{q_i^*}$'s support:

$$\overline{q}_i^* = \left(\frac{\sigma(\xi, t_i)}{-\xi}\right). \tag{1.7}$$

For any fixed types t_i, this upper bound is by definition $\boldsymbol{q_i^*} \overset{(1.6)}{\equiv} \frac{1}{N}\left[\boldsymbol{Q} - c^* \sum_{j \neq i} \boldsymbol{t_j} + (N-1)c^* t_i\right]$ determined by equilibrium coefficient c^*. Since in the equilibrium c^* is strictly positive, i's winning quantity achieves its maximal value when the total supply realizes at its maximum $\overline{Q}$ and all other agents draw the minimal type $\underline{t}$:

$$\overline{q}_i^* \overset{(1.6)}{=} \frac{1}{N}\left[\overline{Q} - c^*(n-1)\underline{t} + c^*(n-1)t_i\right] \text{ with } c^* = \left(\frac{N(1-\xi)-1}{\rho(N-1)(1-\xi)}\right) > 0. \tag{1.8}$$

The scale parameter $\sigma(\xi, t_i)$ must be such that both (1.7) and (1.8) hold. Notice that the lowest amount any type may win in equilibrium $\underline{q}_i^*$ is always 0 because total supply may be 0.

Explaining Bidding Behavior. The bidding function is increasing in the type and strictly decreasing in quantity. To derive an intuition for its functional form for relevant quantities $q \in [0, \overline{q}_i^*]$, I decompose it into three parts. The first is the true marginal valuation of the lowest type $v(q, \underline{t})$, the second, a type-specific mark-up $M(t_i)$ and the third a type-independent shading factor $S(q)$:

$$b^*(q, t_i) = v(q, \underline{t}) + M(t_i) - S(q)$$

with

$$M(t_i) \equiv \left(\frac{1}{1-\xi}\right)(t_i - \underline{t}) \text{ and } S(q) \equiv \left(\frac{\overline{Q} - Nq}{N(1-\xi)-1}\right)(-\xi)\rho \text{ for } q \in [0, \overline{q}_i^*].$$

Consider first the behavior of the lowest type. Since the distribution of types is common knowledge, the lowest type has no private information. Everybody knows that everyone

[11] Recall that for large amounts the agent submits his true marginal valuation which is never negative by construction.

must at least draw a type of size $\underline{t}$. This agent submits his true marginal willingness to pay $v(q, \underline{t})$ in addition to a shading factor $S(q)$. This factor determines the amount by which his shading differs across quantities (differential bid shading). Such strategic demand reduction is optimal because the true marginal willingness to pay is not constant but strictly decreasing, $\rho > 0$. Notice that this factor is independent of the type, which was one of the conditions under which strategizing in pay-as-bid auctions is similar to bidding in first-price auctions. It is strictly positive for $q = 0$ and 0 for the highest amount, $\overline{q}_i^*$, the lowest type might win.

In contrast, an agent who draws a higher type than the lowest one, values each unit of the good more. He should bid a higher price. If he bid truthfully, he would submit a mark-up of $(t_i - \underline{t})$ for each amount. Since his information is private, however, he does not bid his full extra valuation, but only a fraction $\left(\frac{1}{1-\xi}\right) \in \left(0, \frac{1}{2}\right]$ of it. This fraction depends on $\xi \leq -1$. It assumes its maximal value of $\frac{1}{2}$ when the winning quantity is uniformly distributed, which is the case for $\xi = -1$. As ξ decreases it approaches 0. Just as the lowest type any other agent shades his bids differently across quantities because the true marginal valuation is strictly decreasing. At the highest winning quantity, the shading factor is strictly positive for any $t_i > \underline{t}$. This means that any type higher than the lowest shades the largest amount he might win in the symmetric equilibrium by a type-specific discount in addition to an amount coming from differential bid shading.

Interestingly, behavior of privately informed bidders is similar to behavior of bidders without private information. This is easy to see when comparing my example to previous work by Ausubel et al. (2014) (see Appendix A.3.1 for details). They derive the unique linear equilibrium in a pay-as-bid auction in an environment in which agents are only uncertain about the total amount that will be for sale (Proposition 7). As it turns out, agents with private types bid like symmetrically informed agents who all draw the same type $\underline{t}$, just adding the type-specific mark-up $M(t_i)$. All strategic incentives that come from agents having private information are captured by this mark-up.

To close the article I come back to its main theme. Regarding the comparison of bidding in pay-as-bid and first-price auctions, the example illustrates the usefulness of my bid-representation for pay-as-bid auctions. Without it, the similarities of shading behavior across auction formats, summarized in my main results, is difficult to see. This is because bidding functions typically look extremely different, even when the agent's type in the first-price auction is drawn from the same distribution as i's winning quantity in the pay-as-bid auction. Those are the two random variables that must be compared to understand the connection (as explained in Section 1.2). Under the uniform distribution, for instance,

$$\beta^*(s) = \left(\frac{N-1}{N}\right) s \qquad \text{for } s \in \left[0, \overline{S}\right]$$

$$b^*(q, t_i) = \left(\frac{1}{2}\right) [t_i + \underline{t}] - \left(\frac{\rho}{2N-1}\right) \left[(N-1)q + \overline{Q}\right] \qquad \text{for } q \in [0, \overline{q}_i^*]$$

where $\overline{q}_i^*$ could be normalized to match $\overline{S}$, the two bidding functions have not much in common. By comparing them one would not come to the conclusion that in a pay-as-bid auction each bidder shades his bid for 1 of N shares as if he competed with $(N-1)N$ bidders in a first-price auction with independent private types (Main Result).

1.5 Conclusion

Recent literature suggests that strategic incentives in multi-unit auctions differ from those in single-unit auctions when bidders demand more than one unit. It has been shown that bidders shade their bids differentially across quantities when they have multi-unit demand. Such strategic behavior is not present in single-unit auctions and was taken to be the reason for which analogies between single- and multi-unit auctions break down. I refine this view and highlight the importance of the type of uncertainty bidders face. Bidding behavior in a pay-as-bid share auction with symmetrically informed bidders that are uncertain about the total amount for sale, is actually analogous to bidding in the first-price auction: Each of N bidders shades his bid for 1 of N shares as if he competed with $(N-1)N$ bidders for an indivisible good in a first-price auction. This observation can generalize to an environment in which bidders are only ex-ante symmetric, each drawing an iid private type. However, pay-as-bid auctions seem strategically more complex than first-price auctions when agents flatten their bidding functions for small amounts, or shade bids not only differently across quantities, but across types.

Future work could concentrate on analyzing bidding behavior with private information. With our poor knowledge of how equilibria could look like, we know extremely little about how bidders behave in one of the most commonly used auction formats to allocate assets and commodities for high stakes. The complication arises because equilibrium bidding functions are, except in very rare exceptions, non-linear in quantity. This is an important difference to the other most commonly used multi-unit auction format, the uniform-price auction and might be the reason for which we have a much better understanding of bidding in uniform-price auctions relative to pay-as-bid auctions. In addition to theoretic value, a complete characterization of bidding strategies in pay-as-bid auctions where bidders are not symmetrically informed would be useful for the related empirical literature (in Industrial Organization) with recent work by Hortaçsu et al. (2018), Allen et al. (2018) and others. These papers estimate the true valuations of bidders in multi-unit auctions. A typical goal is to perform a counterfactual analysis to find out how much could be gained when changing the rules of the auction, for example, by introducing a reserve price. Lacking a one-to-one mapping between (estimated) true valuations and (counterfactual) bidding choices, makes it difficult to achieve this goal. My bid-representation theorem might be useful to determine such mapping.

Chapter 2

Interconnected Pay-As-Bid Auctions

2.1 Introduction

Pay-as-bid auctions, also known as discriminatory price auctions, are used globally to allocate large classes of assets and commodities. In these auctions participants submit bid schedules demanding different quantities at different prices. Their individual demands are aggregated to determine the market clearing price. Each bidder wins the amount he asked for at the clearing price and pays as he bid for all units he won. In the market for government securities, dozens of countries including China, Japan, Germany, France, India, Brazil and Canada use this format to allocate government debt to those willing to purchase it.[1] Outside of the financial sector, the pay-as-bid auction distributes electricity generation and allocates emission credit in several countries, summing overall to a transferred volume of trillions of dollars each year.[2] Although many of these auctions are held regularly, some even daily, their analysis has thus far been on a stand-alone basis. Yet, pay-as-bid auctions hardly ever take place in isolation; they are interconnected. Influential bidders oftentimes participate in not just one but several of these auctions. One good example is that of Treasury auctions. They are held on a regular basis to the same group of financial institutions, the primary dealers. The primary dealers typically form the largest bidder group and are obligated to participate in the majority, if not all of their nation's Treasury auctions to keep their dealer status. In addition, global financial institutions buy Treasury bonds of different countries. In 2014, for instance, big banks such as Deutsche Bank, Commerzbank, Barclays and HSBC engaged in all Treasury auctions held by Belgium, France Germany, Italy, the Netherlands and Spain to purchase more or less substitutable government bonds of different countries or maturities (Beetsma et al. (2015)). Oftentimes, different Treasury auctions take place in parallel. Since there are more issuers of government securities than there are days in the week, Treasury auctions by different countries are regularly run on the same day.[3] Furthermore, governments of large economies like Canada, Brazil, France, Japan, China, and the US, sell bonds

[1] According to a recent survey by Brenner et al. (2009) 33 out of 48 surveyed issuers of government bonds use the pay-as-bid auction. The US used the pay-as-bid auction from 1929 until 1992 excluding short intervals of experimentation. It formally switched to the uniform-price auction in 1992.

[2] See Brenner et al. (2009), Bartolini and Cottarelli (1997), Ghazizadeh et al. (2007), Maurer and Barroso (2011).

[3] The following calendar gives a nice overview of EU countries (updated on 10/07/2018): `https://europa.eu/efc/eu-wide-indicative-issuance-calendar_en`

of different maturities in separate parallel auctions.[4] In total, a high volume of identical or slightly different Treasury notes, bills and bonds is transferred in simultaneous multi-unit auctions to the same group of bidders. How should agents bid in such interconnected pay-as-bid auctions? Can sellers exploit the interlinkage across auction markets so as to increase revenues; and is it a good idea to issue highly substitutable securities in separate, parallel auctions instead of selling all in one auction?

Providing answers to these question is not possible when considering an auction that is linked to another one in isolation. Doing so, one neglects relevant strategic incentives that are propagated by such interconnection. Neither can one give well-founded recommendations to bidders on how to bid optimally, nor to sellers on how to raise higher revenues. To study how strategies and policy recommendations change when taking into account potential spill-over effects across auctions, this paper introduces a framework of two simultaneous pay-as-bid auctions. Each auction offers a perfectly divisible good to the same group of symmetric bidders. They are uncertain about the total amount that will be for sale and purchase shares of both goods (Wilson (1979)). The auctions may be interlinked through both sides of the market: the demand side because auction participants may value the goods as substitutes; the supply side because the total amount for sale may be correlated across markets. I show that there exists a unique symmetric Bayesian Nash Equilibrium when total supply is drawn from a distribution that assumes weakly decreasing hazard rates, and derive its functional form (Theorem 3). I then develop practical policy recommendations on how to exploit the interconnection across auctions to increase revenues. First, by choosing the type of good that is sold in parallel. It may be more or less substitutable. Secondly, by manipulating the correlation of total supply across auctions.

In practice the auctioneer can set this correlation directly if he is allowed to adapt the amount for sale over the course of the auction, so for instance in Treasury auctions by Germany, Greece, Belgium, Turkey or Sweden (Brenner et al. (2009)). In countries in which total supply must be announced before the auction and cannot be changed, the seller can typically influence the correlation of supply that goes to (regular) bidders indirectly thanks to non-competitive tenders. Non-competitive tenders only specify a quantity that will be purchased at either the average price paid by (regular) bidders or the market clearing price. How many non-competitive tenders there are and how many of them will be served is unknown to the (regular) bidders and may be affected by the auctioneer.

To increase revenues sellers should offer independent goods. When goods are substitutes, total supply should be unequal in size with high probability. Negative correlation of supply quantities fosters competition among bidders. Interestingly, selling identical goods, or more

[4] Given the broad variety of financial securities through which it finances its various operations worth a huge volume of bills and notes each year, it can be very inconvenient and time consuming to sell all the different types of bonds sequentially. Simultaneous sales provide an easy solution, in particular for large economies. All of the named countries but the US run pay-as-bid auctions. The US Treasury runs uniform-price auctions.

loosely speaking, perfect substitutes of negatively correlated supply in two separate parallel auctions generates higher revenues than offering all in one integrated auction. This finding provides a rationale for federal governments to hold Treasury auctions of identical or very similar bonds and bills in parallel, provided the implementation costs of each auction are sufficiently low. As such, it could help to explain why so many Treasury auctions of highly liquid bills take place at the same time - a common practice which seems odd from the perspective of mechanism design.

While this article is purely theoretic, it contributes to a related empirical literature. Following Hortaçsu (2002) and Kastl (2011), researchers recover the willingness-to-pay that rationalizes each observed bid building on necessary conditions for equilibrium behavior of stand-alone auctions. A discretized version of the necessary condition I derive in this article is ideal for such structural estimations. It allows Allen et al. (2018) to quantify the extent to which the demands for government securities of different maturities are interdependent in the primary market of Treasury bills, using data from Canadian Treasury auctions.

The remainder of the article is structured as followed. The subsequent section reviews the related literature. Having set up the model in Section 2.3, Section 2.4 focusses on the equilibrium analysis, while Section 2.5 discusses how bidding and revenue depends on the interconnection across acutions. Section 2.6 concludes. All proofs are given in Appendix B.

2.2 Related Literature

Two lines of research are most closely related to my work. The first is the literature on isolated multi-unit auctions with perfectly divisible goods, so called share auctions. It builds the methodological fundament for my model of interconnected share auctions. Share auctions were introduced by Wilson (1979) and further developed, most notably, by Back and Zender (1993), Wang and Zender (2002).[5] Most literature of share auctions concentrates on the uniform-price auction (Klemperer and Meyer (1989), Vives (2011) and Rostek and Weretka (2012)). It is the direct competitor of pay-as-bid auction, and the second of the two most frequently used multi-unit auction formats. In fact, surprisingly little is known for (stand-alone) pay-as-bid auctions. Except in in special circumstances the literature still struggles "to even compute best-response strategies" (Woodward (2015), p.1). My framework allows me to shut down the interconnection between markets. In doing so, my model is very similar to Pycia and Woodward (2017) who build on Ausubel et al. (2014) and others. We all rely on the simplifying assumption that bidders are symmetrically informed in that they all share the same type. This assumption is also the starting point in Wittwer (2018c) which highlights differences and similarities between pay-as-bid and first-price auctions.

The second line of research consists of few contributions that go beyond the analysis of stand-alone auctions in taking potential interlinkages across auctions into account. In this

[5] Smith (1966) was the first to discuss a framework which can be interpreted as a divisible-good model.

regard the literature on single-unit auctions is more advanced than the literature on multi-unit auctions: Krishna and Rosenthal (1996), Rosenthal and Wang (1996), Jeitschko and Wolfstetter (2002), Menezes and Monteiro (2003), Jofre-Bonet and Pesendorfer (2014) and others analyze environments in which substitutes or complements are auctioned simultaneously or sequentially in several single-unit auctions to so called global bidders. None of them concentrates on the question of how bidding changes in the degree of substitutability or complementarity. Oftentimes the modeling set-up does not easily allow for such analysis (e.g. Jofre-Bonet and Pesendorfer (2014)).
In the literature of multi-unit auctions, interconnections across auction markets have been neglected so far, with few exceptions. Wittwer (2018a) is most closely related to this article; yet both the auction format and research focus differ. Wittwer (2018a) derives an equilibrium for simultaneous uniform-price double auctions each offering one good. The main finding is an irrelevance result that tells us under which conditions such a disconnected market design generates the same allocation as a connected market that allows both goods to be traded in one single (multi-object) uniform-price double auction. Furthermore, the literature on energy procurement auctions studies environments in which a homogeneous commodity (electricity) is produced and consumed in local markets that are connected through transmission networks (e.g. Wilson (2008)), Holmberg and Philpott (2012)). Local bidders (firms) supply energy which is traded globally subject to capacity constraints on the transportation channel. This set-up is not fitting for auctions in the financial sector. They differ substantially from electricity procurement auctions. First, bidding firms do not participate in several auctions, but stay in their local market. This is one of the key features of my set-up. Bidders go to several auctions. A second feature is that, unlike electricity, the goods for sale in different auction markets do not have to be the same. The bidders' preferences for the products of different auctions play a key role. Different from the bidding firms in a procurement auction, who do not care in which market their product is consumed, bidders in the financial sector care which good they buy.

Finally, a line of research developed by computer scientists relates to my work. They are interested in quantifying the cost of selling related indivisible objects in separate auctions that are run in parallel, relative to allocating the objects according to some mechanism that achieves the fist-best.[6] Complementary to the economic literature which "generally focus on exact and optimal solutions", they derive "approximation guarantees for equilibria of auctions in complex settings" (Roughgarden et al. (2017), p. 59). Most of their work concentrates on single-unit auctions (e.g. Feldman et al. (2015a)). Two exceptions are Syrgkanis and Tardos (2013) and Feldman et al. (2015b). The first show that m simultaneously run uniform-price auctions achieve "at least" $\frac{e-1}{4e} \approx 0.158$ of the expected optimal effective welfare. The latter let such a market grow large. None of the contributions I am aware of cover simultaneous pay-as-bid auctions.

[6] They do so by computing the "(Bayesian) price of anarchy", pioneered by Koutsoupias and Papadimitriou (1999) and Christodoulou et al. (2011). This price is "the ratio between the worst possible Nash equilibrium and the social optimum" (Koutsoupias and Papadimitriou (1999), p. 404). For more background consider Roughgarden et al. (2017)'s survey.

2.3 Model

I first describe the supply and then the demand side of the model. Throughout random variables are highlighted in **bold**.

Supply Side. Two pay-as-bid auctions, indexed $m = 1, 2$, are run in parallel. Each offers one perfectly divisible good.[7] The total supply quantities of both goods $\{\boldsymbol{Q_1}, \boldsymbol{Q_2}\}$ are drawn from a commonly known, joint distribution $F(\cdot, \cdot)$ with marginal distributions $F_{Q_m}(\cdot)$. All distribution functions are twice differentiable,[8] and assume density functions $f(\cdot, \cdot), f_{Q_m}(\cdot)$ that are strictly positive on bounded support $[0, \overline{Q}_1] \times [0, \overline{Q}_2]$. Building on Pycia and Woodward (2017) one can extend the model to distributions with unbounded support in presence of arbitrary small but strictly positive reserve prices. Those bound the distribution of supply endogenously.

My main theorem does not rely on any particular functional form for the joint distribution of total supply. When analyzing the role of the market's interconnectedness I want to isolate the spill-over effects on the demand from the one on the supply side. The following joint distribution, known as generalized Farie-Gumbel-Morgenstern (FGM) copula, enables me to separate the two effects of interests. It allows for a variation in the correlation parameter (the connection on the supply side) without affecting the marginal distributions, which - as we will see later - determine bidding behavior (and with it the connection on the demand side):[9]

$$F(Q_1, Q_2) = F_{Q_1}(Q_1)F_{Q_2}(Q_2)\left\{1 + \theta(\rho)[1 - F_{Q_1}(Q_1)][1 - F_{Q_2}(Q_2)]\right\} \tag{FMG}$$

where

$$\theta(\rho) \equiv \rho\left(\frac{\sigma_1\sigma_2}{c_1c_2}\right) \text{ such that } |\theta(\rho)| \leq 1$$

given

$$\sigma_m^2 \equiv Var(\boldsymbol{Q_m}) \text{ and } c_m \equiv \int F_{Q_m}(x)[1 - F_{Q_m}(x)]dx > 0.$$

[7] Similar to the frequent assumption that the set of available prices is dense in the literature on single-unit auctions, the assumption of perfect divisibility is a continuous approximation of a discrete set of quantities (Woodward (2015)). With imperfect divisibility of goods or buyers who can submit only a maximal amount of bids the analysis becomes more complex due to discontinuities and rationing. This has been demonstrated recently by Hortaçsu and McAdams (2010) and Kastl (2011, 2012). Earlier contributions with indivisible goods include Katzmann (1995), Engelbrecht-Wiggans (1998), Swinkels (2001), Engelbrecht-Wiggans and Kahn (2002), Lebrun and Tremblay (2003), Chakraborty (2004, 2006), Anwar (2007). Besides, Armantier et al. (2008) and others approximate equilibria for multi-unit auctions.

[8] Twice differetiability is needed for the sufficient condition. The necessary condition hold for distributions which are only differentiable.

[9] Notice that for given marginal distributions, condition $|\theta(\rho)| \leq 1$ puts a restriction on the range of correlation parameters. There are other copula functions that achieve a larger range of correlation. Yet, they are more complex and make the analysis mathematically much more involved without bringing greater insights.

Demand Side. $N \geq 2$ bidders participate in both auctions. They all share the same, potentially two-dimensional type $\boldsymbol{s} \equiv \begin{pmatrix} \boldsymbol{s_1} & \boldsymbol{s_2} \end{pmatrix}$. Unknown to the seller(s), it is drawn from an arbitrary distribution $F_S(\cdot,\cdot)$. From the bidders' perspective the type is just a number. It is kept fixed throughout the article.

Abstracting from asymmetric information across bidders, which may, for instance, come from private information, is a simplifying assumption. First I avoid having to cope with strategic bid-flattening. So called strategic ironing has recently been discovered to play a role in any equilibrium of a private-value stand-alone pay-as-bid auction under weak assumptions (Theorem 2 by Woodward (2016)). It has been ignored in the previous literature and complicates the analysis dramatically. Second, it allows me to solve for a closed form solution of the bidding function analogous to Pycia and Woodward (2017) and Wittwer (2018c). The necessary condition on which it builds, however, generalizes to much more flexible environments, including those with private information. It holds for a very broad class of utility functions.

To understand how bidding depends on the degree of substitutability across auctions, I let

$$V(q_1, q_2) = \sum_{m=1,2} W_m(q_m) - \delta q_1 q_2 \text{ with } \delta \geq 0 \tag{2.1}$$

with strictly concave and twice continuously differentiable (C^2) functions $W_m(\cdot)$. Without loss, I normalize $W_m(0) = 0$. Parameter δ measures the degree of substitutability between the goods. This is best understood by looking at the bidder's true marginal willingness to pay (MWTP) for quantity q_m. It represents the bidder's true inverse demand in auction m and is given by the partial utility the bidder receives from consuming amount q_m conditional on purchasing amount q_{-m} of the other good:

$$v_m(q_m, q_{-m}) = w_m(q_m) - \delta q_{-m} \quad \text{for } m = 1,2; m \neq -m \tag{2.2}$$

with $w_m(q_m) \equiv \frac{\partial W_m(q_m)}{\partial q_m}$. The higher δ, the *less* the bidder is willing to pay for good m, *the more* he has of the other good $-m$. Substitutability increases. Setting $\delta = 0$ one can shut down any interconnection between markets to be back to the case of a stand-alone auction.

Market Clearing & Equilibrium Concept. In each auction each bidder chooses a strictly decreasing and differentiable bidding function. It is an inverse demand specifying a bid price for each possible quantity $q_m \in [0, \overline{Q}_m]$

$$p_{i,m}(\cdot) : [0, \overline{Q}_m] \rightarrow [0, \infty) \quad \text{for } m = 1,2. \tag{2.3}$$

A point on the bidding function will be called marginal bid. It is the bid price offered for a particular amount q_m. Notice that, even though the bidder has preferences over both goods $V(q_1, q_2)$ he cannot purchase both in the same market. By definition of a pay-as-bid auction only units of the *same* good are for sale within one auction. It is therefore impossible to make price offers in market m dependent of quantities for sale in the other market q_{-m}.

In a more general version of the model bidders would be allowed to submit bidding functions that are only weakly decreasing, and might in addition have discontinuities. This increases the mathematical complexity without adding major insights to the policy analysis. Allen et al. (2018) demonstrate how the model extends when bidders submit step-functions. To keep the analysis simple I further disregard participation constraints and assume that all bidders must be active in both auctions. In the example of Treasury auctions, where primary dealers have to participate in all auctions in many countries, for example, in Canada, this is a fitting assumption.

It will be convenient to express the bidding strategies of the other players $j \neq i$ in terms of quantity per bid price (as demand) rather than bid prices per quantity (as inverse demand)

$$x_{i,m}(\cdot) : [0,\infty) \to [0, \overline{Q}_m] \quad \text{for } m = 1, 2. \tag{2.4}$$

Once all bidders have submitted their bids, an auctioneer determines the minimal price at which aggregate demand meets aggregate supply, called the market clearing price, p^c_m. If there is no excess demand each bidder wins the amount he asked for at this price, his clearing price quantity, $q^c_{i,m}$, and pays $\int_0^{q^c_{i,m}} p_{i,m}(x)dx$.

Both the market clearing price and quantity are, for given strategies of all agents, implicitly defined by market clearing. Taken together they constitute a point on the residual supply curve of agent i, $RS_{i,m}(\cdot)$. This curve specifies what is left of the total supply once all other agents than i have demanded their shares. From an ex-ante perspective, it is unknown to the agent as it depends on the random total supply.

Definition 2. *At market clearing he wins amount* $\boldsymbol{q^c_{i,m}}$ *at price* $\boldsymbol{p^c_m}$ *defined by*

$$\boldsymbol{q^c_{i,m}} = RS_{i,m}(\boldsymbol{p^c_m}) \equiv \boldsymbol{Q_m} - \sum_{j\neq i} x_{j,m}(\boldsymbol{p^c_m}) \text{ with } \boldsymbol{p^c_m} = p_{i,m}(\boldsymbol{q^c_{i,m}}). \tag{2.5}$$

The support of i's clearing quantity is denoted $[0, \overline{q}^c_{i,m}]$*, where* $\overline{q}^c_{i,m}$ *solves (2.5) for* $Q_m = \overline{Q}_m$.

Since agents do not know how much they will win, an adequate solution concept is the Bayesian Nash Equilibrium. I restrict my attention to pure-strategy equilibria. They consist of a set of bidding functions $\{p^*_{i,1}(\cdot), p^*_{i,2}(\cdot)\}_{i=1}^N$ that maximize each bidder's expected total surplus (utility net of total payment) from winning the ex-ante unknown clearing price quantities given all others $j \neq i$ choose $\{p^*_{j,1}(\cdot), p^*_{j,2}(\cdot)\}$.

Definition 3. *A pure-strategy Bayesian Nash Equilibrium (BNE) is a set of bidding functions such that*

$$\{p^*_{i,1}(\cdot), p^*_{i,2}(\cdot)\} \in \underset{p_{i,1}(\cdot),p_{i,2}(\cdot)}{\operatorname{argmax}} \mathcal{V}(p_{i,1}(\cdot), p_{i,2}(\cdot)) \equiv \mathbb{E}\left[V(\boldsymbol{q^c_{i,1}}, \boldsymbol{q^c_{i,2}}) - \sum_{m=1,2} \int_0^{\boldsymbol{q^c_{i,m}}} p_{i,m}(x)dx\right] \quad \forall i \in N$$

where $\boldsymbol{q^c_{i,1}}, \boldsymbol{q^c_{i,2}}$ *are the clearing price quantities, implicitly defined by (2.5).*

Given the symmetric environment of this article, in which all bidders have the same type, it is natural to focus on symmetric equilibria where all bidders choose the same pair of functions $\{p_1^*(\cdot), p_2^*(\cdot)\}$ and share the total supply of both auctions equally. Following Wittwer (2018c), it will be convenient to work with the (random) equilibrium winning quantity instead of the total supply: $\boldsymbol{q_m^*} \equiv \frac{\boldsymbol{Q_m}}{N} \in \left[0, \frac{\overline{Q}_m}{N}\right] \equiv [0, \overline{q}_m^*]$ with marginal distribution $F_{q_m^*}(\cdot)$ and density $f_{q_m^*}(\cdot)$ for $m = 1, 2$.

Equilibrium bidding will depend on how much the bidder expects to value one good given he wins the equilibrium amount in the other auction, $\mathbb{E}\left[v_m\left(q_m, \boldsymbol{q_{-m}^*}\right)|q_m\right]$.

Assumption 1. *The expected true marginal willingness to pay for amount q_m*

$$\mathbb{E}\left[v_m\left(q_m, \boldsymbol{q_{-m}^*}\right)|q_m\right] \overset{(2.2)}{=} w_m(q_m) - \delta\mathbb{E}[\boldsymbol{q_{-m}^*}|q_m] \tag{2.6}$$

is (i) strictly decreasing and (ii) non-negative for all $q_m \in [0, \overline{q}_m^*]$ for $m = 1, 2$; $m \neq -m$.

Condition (ii) ensures that no agent wants to offer negative bids for quantities that he can win. Given condition (i) this condition is fulfilled as long as the aggregate valuation of all bidders N for the very first units $v_m(0,0)$ is sufficiently high. Intuitively, it says that the market's total demand for the very first unit must be high enough. Thanks to condition (i), no bidder has an incentive to submit an increasing bidding function in the symmetric equilibrium. Such functions would not be valid by the rules of the auction. The condition always holds when auctions are isolated ($\delta = 0$) because $W_m(q_m)$ is strictly concave by assumption. When they are interconnected, it gives a lower bound of negative correlations ($\delta > 0$) when the joint distributions generates a conditional expectation $\mathbb{E}[\boldsymbol{q_{-m}^*}|q_m]$ that increases monotonically in the correlation coefficient.[10] This is, for instance, the case for the generalized Farie-Gumbel-Morgenstern copula distribution (FMG), where

$$\mathbb{E}\left[v_m\left(q_m, \boldsymbol{q_{-m}^*}\right)|q_m\right] = w_m(q_m) - \delta\left\{\mathbb{E}[\boldsymbol{q_{-m}^*}] - \rho\left(\frac{\sigma_m\sigma_{-m}}{c_m}\right)\left[\frac{1}{N}\right]\left[1 - 2F_{q_m^*}(q_m)\right]\right\}. \tag{2.7}$$

To see why the parameter space needs to be restricted when δ and ρ take opposite signs, consider the case of negatively correlated substitutes ($\rho < 0, \delta > 0$). Due to substitutability, the *true* marginal willingness to pay in market m, $v_m(q_m, q_{-m})$, decreases in q_{-m}. Moreover, the conditional expectation $\mathbb{E}[\boldsymbol{q_{-m}^*}|q_m]$ is decreasing because the supply is negatively correlated. The stronger the negative correlation, the steeper its slope. Together this implies that the *expected* marginal willingness to pay is decreasing only if $|\delta|$ and $|\rho|$ are sufficiently low.

2.4 Equilibrium

The first part of the article derives the symmetric Bayesian Nash Equilibrium. Before stating the main theorem, I explain how agents choose their bids in any, not necessarily symmetric,

[10]Notice, that not all distributions feature a conditional expectation that is increasing in the correlation coefficient. The opposite restriction of the parameters holds if it decreases as the correlation increases. When it is non-monotone in the correlation coefficient, the assumption has no intuitive appeal.

equilibrium. To build an intuition for bidding incentives in simultaneous pay-as-bid auctions it helps to review how bidders behave in an isolated pay-as-bid auction.

Bidding incentives in a stand-alone pay-as-bid auction are similar to those in the well-known first-price auction. In the later, agents only buy one unit of the object, that is, the object itself, placing one bid. In the pay-as-bid auction, they pick several bid prices for several units. When the good is perfectly divisible, a unit becomes a quantity, q_m. Each unit-bid must be chosen optimally. It satisfies a necessary condition, which may be seen as the "multi-unit counterpart of the equilibrium condition for bidding in a first-price auction" (Kastl (2017) p. 7). It describes the trade-off an agent faces when choosing a unit-bid: He "trades off the expected surplus on the marginal (infinitesimal) unit versus the probability of winning it" (Kastl (2017) p. 7). The following Lemma formalizes this trade-off. It is taken from Hortaçsu and McAdams (2018) who themselves builds on Février et al. (2004) and Hortaçsu (2002) with slight modifications to fit the model and notation presented here.

Lemma 1. *Consider an isolated auction of good m, in which agent i is willing to pay $v_m(q_m)$ for amount q_m. A BNE $\{p^*_{i,m}(\cdot)\}_{i=1}^N$ must, for all q_m and $i \in N$ satisfy*

$$[v_m(q_m) - p^c_m] \left(\frac{\partial Pr\left(RS_{i,m}(p^c_m) \geq q_m\right)}{\partial p_m} \right) = Pr\left(RS_{i,m}(p^c_m) \geq q_m\right) \tag{2.8}$$

and clear the market: $p^c_m = p^*_{i,m}(q_m)$.

The right-hand side of optimality condition (2.8), $Pr\left(RS_{i,m}(p^c_m) \geq q_m\right)$, represents the probability of winning the bid. More precisely, it specifies the probability that the bidder wins at least amount q_m when submitting a bid such that $p^*_{i,m}(q_m) = p^c_m$. In other words, it is the likelihood that the market will clear at some price lower than p^c_m. The left-hand side represents the expected surplus on the marginal (infinitesimal) unit. It is the difference between the agent's true marginal valuation for amount q_m and the price he pays p^c_m, $[v_m(q_m) - p^c_m]$, weighted by the probability that the market actually clears at this price, $\partial Pr\left(RS_{i,m}(p^c_m) \geq q_m\right) / \partial p_m$.

Bidding incentives in interconnected pay-as-bid auctions are similar. The key difference is that the agent now participates in two pay-as-bid auctions which offer two perfectly divisible goods. Since these two goods might be substitutable to the agent, the agent's true marginal willingness to pay for amount q_m depends on how much he wins in the equilibrium of the other auction, $q^*_{i,-m}$. As both auctions take place simultaneously, the agent does not know how much he ends up winning of good $-m$ when choosing his bids in auction m. In the optimum, he takes the best guess. In the event of purchasing quantity q_m, he expects a marginal gain of $\mathbb{E}[v_m(q_m, \boldsymbol{q^*_{i,-m}})|q_m]$. Hereby he anticipates that all agents will play the equilibrium strategy in the other auctions.

Lemma 2. *Consider an interconnected auction of good m, in which agent i is willing to pay $v_m(q_m, q_{-m})$ for amount q_m conditional on winning q_{-m}. A BNE $\{p^*_{i,1}(\cdot), p^*_{i,2}(\cdot)\}_{i=1}^N$ must for all $\{q_1, q_2\}$ and $i \in N$ satisfy*

$$[\mathbb{E}[v_m(q_m, \boldsymbol{q^*_{i,-m}})|q_m] - p^c_m] \left(\frac{\partial Pr\left(RS_{i,m}(p^c_m) \geq q_m\right)}{\partial p_m}\right) = Pr\left(RS_{i,m}(p^c_m) \geq q_m\right) \tag{2.9}$$

for $m = 1, 2; -m \neq m$, and clear both markets $p^c_m = p^*_{i,m}(q_m)$.

Notice that in the stand-alone auction $Pr\left(RS_{i,m}(p^c_m) \geq q_m\right)$ is given by the one-dimensional distribution of the good's clearing price. In the interconnected environment it becomes the *marginal* distribution, yet again to be evaluated at p^c_m. Unless auctions are stochastically independent, this marginal distribution must be derived from the *joint* distribution.[11]

In the symmetric framework assumed in this article, where all bidders know their type and only total supply is unknown, all agents split the total amount for sale equally in equilibrium. Each wins $\boldsymbol{q^*_2} \equiv \frac{\boldsymbol{Q_2}}{N}$. In a more general environment, there might be other factors than the total supply that agents are uncertain about. They could, for instance, each have a private type. The optimality condition generalizes to such environments. Neither does it depend on any particular functional form of the utility or the joint distribution of the clearing prices.[12] Thanks to its generality, Allen et al. (2018) use a discretized version of it to estimate the interdependencies across pay-as-bid auctions.

Abstracting from other types of uncertainty allows me to prove the existence of a symmetric equilibrium and derive a closed form solution of the bidding function. In a nutshell, the necessary conditions become a linear differential equation when imposing symmetry across all bidders. They have a unique solution from which no bidder has an incentive to deviate.

Theorem 3. *Let total supply quantities be drawn from a (joint) distribution with weakly decreasing (marginal) hazard rates. There exists a 'unique' symmetric Bayesian Nash equilibrium in which all agents submit*

$$p^*_m(q_m) = \mathbb{E}\left[v_m\left(q_m, \boldsymbol{q^*_{-m}}\right)|q_m\right] + \int_{q_m}^{\overline{q}^*_m} \left(\frac{\partial \mathbb{E}\left[v_m\left(x, \boldsymbol{q^*_{-m}}\right)|x\right]}{\partial q_m}\right) \left[\frac{1 - F_{q^*_m}(x)}{1 - F_{q^*_m}(q_m)}\right]^{\frac{N-1}{N}} dx \tag{2.10}$$

for relevant quantities $[0, \overline{q}^*_m]$ and both goods $m = 1, 2, -m \neq m$.

Similar to Holmberg (2009) the existence of my equilibrium hinges on the assumption that total supply has a (here marginal) distribution with weakly decreasing hazard rate. More recently Pycia and Woodward (2017) have derived a weaker condition for equilibrium existence

[11] The joint distribution specifies the probability that the bidder wins at least quantity q_1 when submitting a bid such that $b^*_{i,1}(q_1) = p^c_1$ in auction 1 and at least quantity q_2 when submitting a bid such that $p^*_{i,2}(q_2) = p^c_2$ in auction 2: $Pr\left(RS_{i,1}(p^c_1)\right) \geq q_1$ and $RS_{i,2}(p^c_2) \geq q_2)$.

[12] To be more precise, it holds for any utility function that is twice differentiable and has continuous cross-partial derivatives, and all distribution functions which are differentiable.

in stand-alone pay-as-bid auctions. Unfortunately their proof does not generalize to simultaneous auctions because it is not possible to solve for the bidder's best reply by maximizing point-wise for each quantity. There are two reasons for why point-wise maximization is no longer feasible when considering simultaneous auctions that are stochastically interdependent. First, the bidder's objective functional of Definition 3, $\mathcal{V}(p_{i,1}(\cdot), p_{i,2}(\cdot))$, which can be expressed as $\int_0^{\overline{Q}_1}\int_0^{\overline{Q}_2} \mathcal{F}(q_1, q_2, p_{i,1}(q_1), p_{i,2}(q_2))dq_1dq_2$, is not additively separable in $p_{i,1}(q_1)$ and $p_{i,2}(q_2)$. It is not possible to determine optimal bidding functions separately in first fixing q_1 to find $p_1^*(q_1)$ and then fixing q_2 to find $p_2^*(q_2)$. Second, since the bidding function in auction 1 cannot depend on the amount of auction 2, one cannot fix both q_1 and q_2 and optimize point-wise over $p_{i,1}(q_1)$ and $p_{i,2}(q_2)$. Instead of point-wise maximization one must maximize over the entire functions. As this article focuses on deriving policy recommendation rather than providing the most general theory of interconnected pay-as-bid auctions I leave it for future work to specify weaker existence conditions than supply distributions with weakly decreasing hazard rates.

The strictly decreasing bidding function (2.10) is unique on the set of relevant quantities $q_m \in [0, \overline{q}_m^*]$. Higher amounts are infeasible in equilibrium, and are therefore uninteresting. Bidders may choose any bid prices for those high amounts as long as their bidding function is differentiable and decreasing on the whole domain $[0, \overline{Q}_m]$. One intuitive possibility is the case where all behave truthfully for infeasible amounts:

$$p_m^*(q_m) = \mathbb{E}\left[v_m\left(q_m, \boldsymbol{q^*_{-m}}\right)|q_m\right] \text{ for } q_m > \overline{q}_m^*.$$

When $\delta = 0$ goods are independent agents behave like in a stand-alone auction selling good m in which they each have a true marginal willingness to pay of $w_m(q_m)$:

$$\delta = 0: \ \mathbb{E}\left[v_m\left(q_m, \boldsymbol{q^*_{-m}}\right)|q_m\right] \overset{(2.6)}{=} w_m(q_m).$$

The equilibrium bidding function coincides with Pycia and Woodward (2017). It is, in addition, analogous to Holmberg (2009)'s equilibrium in a procurement auction where bidders are firms with non-decreasing marginal costs (here decreasing valuations) and submit upward sloping supply functions (here decreasing demand functions). Because a bidder pays as he bids he understates his true marginal willingness to pay for each quantity point q_m that he might purchase in equilibrium $w_m(q_m)$. This is similar to an independent private-value sealed-bid first-price auction, where bidders shade their true types (see Wittwer (2018c) for more details).

When $\delta \neq 0$, the bidder's marginal willingness to pay for q_m depends on how much he will purchase of the other good in equilibrium: $v_m(q_m, \boldsymbol{q^*_{-m}})$. The problem is that he *does not know* how much he receives in the other market because both transactions take place simultaneously. He makes the best guess, exploiting the existing correlation between the random supply quantities across auctions

$$\delta \neq 0: \ \mathbb{E}[v_m(q_m, \boldsymbol{q^*_{-m}}|q_m)] \overset{(2.6)}{=} w_m(q_m) - \delta\mathbb{E}[\boldsymbol{q^*_{-m}}|q_m].$$

In what follows I analyze how the connection across auctions influences bidding and affects expected revenues.

2.5 On the Interconnection between Auctions

Recall that there are two types of interlinkages. The first comes from the demand side of the market. It is generated by the bidders' preferences for the goods. They may be seen as substitutes (δ). The second is on the supply side. It is determined by the correlation (ρ) between the total supply offered in each auction. To isolate the second effect, I specify the joint distribution as generalized Farie-Gubmbel-Morgenstern copula (FMG). It relates any two marginal distributions by a single parameter that reflects correlation and therefore allows me to alter ρ without affecting anything else. To stress the dependence on the two parameters of interest, δ, ρ, I refer to the bidding function as $p_m(\cdot, \delta, \rho)$ throughout this section.

2.5.1 Bidding in Simultaneous Auctions

To analyze how bidding behavior responds to changes in δ and ρ, I decompose the bidding function into two parts: one which is invariant to changes in either parameter, and one that varies $D_m(q_m, \delta, \rho)$. The invariant term coincides with the equilibrium bidding function of a stand-alone auction with true MWTP of $w_m(q_m)$.

Corollary 2. *Let $p_m^{SA}(\cdot)$ be the equilibrium bidding function of an isolated pay-as-bid auction in which each agent is willing to pay $w_m(q_m)$ for amount q_m.*
In the symmetric equilibrium of an interconnected pay-as-bid auction, each agent submits

$$p_m(q_m, \delta, \rho) = p_m^{SA}(q_m) - D_m(q_m, \delta, \rho)$$

with

$$D_m(q_m, \delta, \rho) \equiv \delta \left\{ \mathbb{E}\left[\boldsymbol{q^*_{-m}}\right] + \rho \left(\frac{\sigma_m \sigma_{-m}}{c_m} \right) \left[\frac{1 + 2(N-1) F_{q_m^*}(q_m)}{N(2N-1)} \right] \right\}.$$

Notice that the bidding function is - for the vast majority of marginal distributions - not linear even when the true marginal willingness to pay is linear. This property is already known from isolated pay-as-bid auctions. Here the equilibrium is linear if and only if supply is distributed according to the generalized pareto distribution (Ausubel et al. (2014)). When auctions are interconnected ($\delta \neq 0$), the bidding function is non-linear even for many marginal distributions within this class of distributions. This is because most distributions assume non-linear conditional expectations, $\mathbb{E}[\boldsymbol{q^*_{-m}}|q_m]$, which in turn leads to a discount factor that is non-linear. From the formula, we see that $F_{q_m^*}(q_m)$ has to be linear for $D_m(q_m, \delta, \rho)$ to be linear. This is the case for very few exceptions, as, for example, the uniform distribution. It gives rise to linear bidding functions in isolated and interconnected pay-as-bid auctions.

[12]Schucany et al. (1978) relate parameter θ to the correlation parameter. They define c_m (which they call δ_j) in expression (4) on p. 650 as $c_m = -\int Q_m[1 - 2F_{Q_m}(Q_m)]dF_{Q_m}(Q_m)$ and show it is strictly positive. Crane and van der Hoek (2008) implicitly express c_m as $c_m = \int F_{Q_m}[1 - F_{Q_m}]dQ_m$ when deriving equivalent formulas for the condition expectation on p. 56.

The two parameters that measure the auctions' interconnectedness, δ and ρ, influence bidding behavior only via the discount factor $D_m(q_m, \delta, \rho)$. To understand how bidders react to more closely connected auction markets we consequently must understand how it depends on these parameters. The higher it is the lower the marginal bid.

Corollary 3. *Let* $\delta > 0$.
(i) $D_m(q_m, \delta, \rho) > 0$, *(ii)* $D_m(q_m, \delta, \rho) = \delta \mathbb{E}[\boldsymbol{q^*_{-m}}]$ *for* $\rho = 0$,
(iii) $\frac{\partial D_m(q_m,\delta,\rho)}{\partial \delta} > 0 \; \forall \rho \neq 0$, *(iv)* $\frac{\partial D_m(q_m,\delta,\rho)}{\partial \rho} > 0$, *(v)* $\frac{\partial D_m(q_m,\delta,\rho)}{\partial q_m} > 0$ *if* $\rho > 0$, < 0 *if* $\rho < 0$.

Intuition suggests that competition weakens when bidders have access to another market that offers a substitute. Indeed, marginal bids for substitutes are discounted more strongly relative to a stand-alone auction (statement (*i*)). The closer the substitutes, the less aggressive the bidding (statement (*iii*)). The higher the likelihood of equally sized total supply quantities the weaker the competition among bidders (statement (*iv*)). When supply is independently drawn across markets ($\rho = 0$), the bidder discounts a bid for a particular quantity of good m precisely by how much he expects to value the good less because it is a substitute to the one for sale in the other market (statement (*ii*)). The bidding function shifts downward in parallel. More interesting is the analysis when auctions are stochastically connected ($\rho \neq 0$). Now the bidders no longer simply discount each of their marginal bids by their expected marginal loss or gain in utility. Instead, they exploit the existing correlation between supply quantities to reduce the uncertainty that they are facing. This generates bid shading that differs across quantities. The bidding function no longer simply shifts downward but changes its slope (statement (*v*)). When competing for positively correlated substitutes, bidders submit a steeper bidding function than they would in a stand-alone auction. Two factors drive this behavior: When the bidder wins a lot in one auction he is likely to win a lot also in the other auction since supply is positively correlated. The more he wins of the substitute good in the other auction the less he values the amount in the auction at hand. As a result, the bidder bids less aggressively for higher quantities. The opposite holds when supply is negatively correlated.

2.5.2 Policy Recommendations

So far I have analyzed the auctions from the view point of the bidders. We have seen how they bid optimally in interconnected pay-as-bid auctions and react to changes in the environment. A seller can anticipate such bidding behavior. Can he thereby raise higher revenues?

In equilibrium, a seller always sells the entire quantity he puts up for sale. To increase revenues he therefore only cares about the prices bidders pay, not the quantity they demand. Taken the market design as given, he has two possibilities to exploit the feature that his auction is interconnected to another auction to abstract more rent from the bidders. First, he can choose the type of good he puts up for sale. If possible, he should offer an independent good. More generally revenue increases the lower the degree of substitutability as higher competition among bidders drives up their marginal bids ((*iii*) of Corollary 3). Secondly, he can coordinate his decision on how much to sell with the seller of the other auction. If he himself organizes both auctions, such coordination is not even needed. To increase revenues supply should be negatively correlated ((*iv*) or Corollary 3). Put differently, the amount for sale should be uneven in size with high probability.

Policy Recommendation 1.

(a) Offer independent goods to foster competition across auctions.

(b) When goods are substitutes, supply should with high probability differ in size: The lower the correlation of supply quantities, the higher the revenue in each auction.

Provided issuers of government securities know the degree of substitutability across goods these recommendations can easily be put into practice. Consider the example of Treasury auctions. Say two securities which differ more strongly in terms of risk and maturities are less substitutable to one another than two that are more similar. According to recommendation (a) governments should sell the 12-week bills in parallel with the 30-year bonds rather than with the 36-week bills to raise the necessary funds at an overall lower cost. This is not in line with common practices of most governments. They tend to cluster securities which are similar.[13] The US, for instance, sells the most liquid securities (bills) always separately from securities with longer maturities (notes and bonds).[14] Similarly, neither France nor Canada mix across security categories. While there might be reasons for this policy that my model abstracts from, such as promoting more favorable price dynamics in the secondary market, my work suggests a revision of such practices.

In addition, the government could increase higher revenues by issuing total supply quantities that are negatively correlated (recommendation (b). In countries in which the auctioneer has the right to adapt the total amount for sale during the auction, this recommendation can be implemented directly by drawing from a joint distribution that features a negative correlation. When the total supply is known to all bidders, and cannot be changed during the auction, the seller must find an indirect way to take the recommendation to practice. One way is to stimulate the participation of non-competitive tenders in one auction and discourage it in the other. Recall that non-competitive tenders specify a quantity that is won with certainty. From the perspective of a regular, competitive bidder this would have the same effect as fine-tuning the correlation parameter directly. He only cares about the total amount for sale net of non-competitive tenders, which would with high probability differ in size.

Perfect Substitutes. When two auctions sell the same good, sellers could go one step further than coordinating. They could join forces and organize one big pay-as-bid auction. By definition, a pay-as-bid auction sells units of the same good. Without changing

[13]Japan is an exception. It regularly sells T-bills together with long-term bonds (see `https://www.mof.go.jp/english/jgbs/auction/calendar/index.htm`, updated July 2018)

[14]See the tentative auction calendar for 2018 under `https://www.treasury.gov/resource-center/data-chart-center/quarterly-refunding/Documents/auctions.pdf`

the auction design such integration is therefore only possible when goods are identical, or more loosely speaking perfectly substitutable. Coming back to my motivating example of Treasury auctions, European countries could issue a single European government bond in a single Treasury auction. Furthermore, countries like France or Canada could combine their simultaneous, but separate auctions for bonds with the same or very close maturity dates.

To find out whether integrating both markets increases revenues, I compare bidding behavior in two separate auctions, each selling the same good, with behavior in one integrated auction. To do so, I adapt the framework as follows: First I make the two goods identical by specifying

$$V(q_1, q_2) = \sum_m \left[s - \frac{\delta}{2} q_m \right] q_m - \delta q_1 q_2. \tag{2.11}$$

Now either partial utility $v_m(q_m, q_{-m}) = s - \delta q_m - \delta q_{-m}$ decreases in good m by the same amount it decreases in the other good. Moreover, to create perfectly symmetric market conditions, I let the supply of each of the two separate auctions be drawn from the same marginal distribution $F_Q(\cdot)$ on $[0, \overline{Q}]$. The combined auction sells twice as much of the good. Its total amount for sale is drawn from a distribution $F_{2Q}(\cdot)$ on support $[0, 2\overline{Q}]$. It is identical to $F_Q(\cdot)$ just stretched out over the larger support.

With these changes to the set-up, consider first the separate auctions. As a starting point assume that the total supply quantities are perfectly positive correlated. Since both markets are ex-post identical, a bidder expects to win q also in the other auction conditional on observing q: $\mathbb{E}[\boldsymbol{q^*_{-m}}|q] = q$. According to Theorem 3, the bidder submits

$$p(q) = s - 2\delta q - 2\delta \int_0^{\overline{q}^*} \left[\frac{1 - F_{q^*}(x)}{1 - F_{q^*}(q)} \right]^{\frac{N-1}{N}} dx$$

for $q \in [0, \overline{q}^*]$ in the symmetric equilibrium in both auctions. In comparison, consider the integrated auction, which sells double the amount of the good all in one auction. Since there is only one market that offers the good, there is no need to differentiate where the amount is purchased: $q_1 = q_2 \equiv q$. The total utility from purchasing $q + q = 2q$ is $V(q, q) \overset{(2.11)}{=} 2sq - 2\delta q^2$. The bidder's true marginal willingness to pay for this amount is $\frac{\partial V(q,q)}{\partial q} = 2s - 4\delta q$. From Theorem 3 with $\delta = 0$ it follows that he bids

$$p^{IA}(2q) = 2s - 4\delta q - 4\delta \int_0^{2\overline{q}^*} \left[\frac{1 - F_{2q^*}(2x)}{1 - F_{2q^*}(2q)} \right]^{\frac{N-1}{N}} dx$$

for $2q \in [0, 2\overline{q}^*]$. Since, by construction, the integrals of both bidding functions are identical $p^{IA}(2q) = 2p(q)$. On the aggregate, bidders thus submit the same price in the two separated auctions as they do in the integrated market. The following corollary summarizes.

Corollary 4. *In two separate auctions which offer supply quantities of the same good that are perfectly positive correlated, agents bid as if they competed in just one single auction that sells all at once. They split their bid offers equally across markets.*

For the sellers, this result has an important implication: For any fixed realization of supply, they achieve as much revenue in an integrated auction as they do on the aggregate in two separate auctions given supply is perfectly correlated. From Corollary 3 (*iv*) we know that the revenue of each of these separated but interconnected auctions can be raised by decreasing the correlation of total supply. Together this implies that sellers can do strictly better when leaving the auctions separated as long as they do not offer the same total supply with certainty. Integrating two auctions into one turns out to be revenue decreasing.

Policy Recommendation 2. *Selling a perfectly divisible good in two separate pay-as-bid auctions leads to higher revenue than selling all units of the good in one single auction, as long as the total amounts for sale are not perfectly positively correlated across auctions.*

This is good news for many issuers of government securities. It provides a rationale to offer highly substitutable securities at the same time, yet in separate parallel auctions. Those securities could to be the most liquid bills that only differ slightly in maturity dates. They are already issued in parallel auctions in many large economies.

2.6 Conclusion

This article introduces a model of interconnected pay-as-bid auctions, which simultaneously sell each a perfectly divisible good to the same group of bidders, who are uncertain about the total supply for sale in each auction that is issued at random. I focus on two sources of interconnection. The first comes from the bidders' preferences for the two goods for sale; they might view them as substitutes. The second is on the supply side of the market. It is determined by the correlation between the total supply across auctions. I show that marginal bids for substitutes decrease the higher the substitutability across auctions or the more positive the correlation between supply. To raise higher revenues, sellers should offer independent goods and issue uneven sizes of total supply with high probability when goods are substitutes. Auctioneers of identical goods should split its total supply and sell it in parallel auctions to foster competition among bidders. Taken into real life, this finding provides a rationale for governments and central banks to hold parallel auctions of closely substitutable securities.

There are multiple avenues for further research both theoretically and empirically. One could study how the pay-as-bid auction compares the uniform-price auction. My analysis generalizes without complications to the other most popular multi-unit auction format. Given that the literature on stand-alone auctions has not found a consensus on which auction format is superior, a comparison of the two in presence of substitutes could bring valuable insights. Finally, the model can be taken to the data so as to structurally estimate its parameters, or back out the bidders' marginal valuations. Allen et al. (2018) use a discretized version of the necessary condition presented above to quantify interdependencies in the demand for government securities with data from Canadian Treasury auctions.

Chapter 3

Identifying Dependencies in the Demand for Government Securities

with Jason Allen[1] and Jakub Kastl[2]

3.1 Introduction

The objective of the Office of Debt Management (ODM) of the U.S. and of the corresponding offices in most countries is to achieve the "lowest cost of financing over time." In order to fulfill this objective, the ODM has to decide how to sell the debt: what format to use, which securities to offer and how to allocate a given target amount across different securities. Since the securities are clearly not independent, it is important to evaluate the full demand system when making such decisions. Previous literature, discussed further below, has addressed the issue of the choice of format. The focus of this paper is to propose a method for identifying the dependencies of demands across different securities in order to help policymakers make a more informed decision on how to split the supply across securities of different maturities.

A central issue that arises when estimating demand systems is unobserved heterogeneity: how to make sure that variation in quantity choices is attributable to variation in prices and not something omitted that is correlated with price, for instance quality in case of a typical discrete choice model. This is usually addressed by employing instruments aimed at isolating such exogenous variation by making the appropriate exogeneity and validity assumptions. We, instead, utilize a particular institutional feature that is surprisingly common in auctions of government debt such as those run by the US, Japan, Brazil, France, China and Canada: different securities (i.e., Treasury bills and bonds of different maturities) are sold simultaneously in parallel auctions. We extend previous results on identifying willingness to pay from bidding data in auctions to allow for the willingness to pay to depend on not only the allocation of the underlying security, but also on holdings of securities of other maturities. This setup enables us to estimate a full demand system in the primary (i.e., auction) market, allowing for any potential substitution patterns across securities, including complementarities.

[1]Bank of Canada: The presented views are those of the authors, not necessarily of the Bank.
[2]Princeton University, NBER and CEPR

More specifically, we use data on all 3, 6, 12-month Canadian Treasury bill auctions from 2002 to 2015 to estimate a model of simultaneous pay-as-bid auctions. We extend recent techniques for estimating bidders' marginal willingness to pay in individual multi-unit auctions developed by Hortaçsu (2002) and Kastl (2011) who build on the pioneering work of Guerre et al. (2000).[3] The related theoretic literature is more advanced in this regard (e.g. Wittwer (2018a,b)). Typically, multi-unit auctions are held simultaneously, or sequentially, but over a very short time-span. Valuations that the auction participants attach to the different securities are, among other things, a function of the options available in the parallel auctions. These valuations, therefore, should be estimated jointly.

We find non-negligible interdependencies in the bidders' demand for government securities. The most liquid security (the 3-month bill) is complementary to either of the two longer maturities (the 6 and 12-month bill). We estimate that the marginal valuation for a 3-month T-bill increases when going from an allocation with no other T-bills of other maturities at all to an allocation corresponding to the average observed allocations of 6 and 12-month bills (about 200 million each) by about 0.5 basis points. Seemingly small in absolute size this "cross-market" effect is actually relatively large compared to the "own-market" effect: the marginal value for the 3-month bill drops by about 1 basis point when going from none to 400 million of these bills. Moreover, financial institutions who participate in the auctions for longer maturities exhibit a shift in preferences over the course of our sample period (2002-2015). During the global financial crisis (2007-2010) the 3-month bill was a substitute for bidders in the 6-month as well as the 12-month auction. Going from none to the observed average allocation of the 3-month bill decreases the bidder's willingness to pay in the 6/12-month auction by about 0.5/1.2 basis points, respectively. However, after the crisis we find that they are no longer substitutable. In the most recent years (2011-2015) all maturities have become complementary for bidders in all auctions with one exception. For participants in the 12-month auction, the 3-month bill behaves essentially as an independent good. Compared to the period before the crisis, the degree of complementarity has increased across all maturities.

This result may seem surprising. Securities that are highly similar in terms of risk are typically classified as substitutes. Dating back to the 1960s, there has been a long lasting debate in monetary economics about the degree of substitutability of "cash-like" assets. One example is Sertelis and Robb (1986). They estimate the degree of substitution between Canadian monetary assets (namely narrow money, other checkable deposits, savings and time deposits) in a model with a money-in-the-utility function. In line with most articles of this early literature, they find a low degree of substitution.[4] The more recent financial literature most

[3]Using economic models, researchers recover the marginal willingness to pay that rationalizes each observed bid (e.g. Février et al. (2004), Armantier and Sbaï (2006), Hortaçsu and McAdams (2010), Hortaçsu and Kastl (2012), Cassola et al. (2012), Hortaçsu et al. (2018)). These papers then proceed to analyze whether the auction mechanism is revenue-maximizing, whether it leads to an efficient allocation, or to address other economic questions of interest, such as quantifying the information advantage of observing the order flow of customers.

[4]See Feige and Pearce (1977) for an early survey on "the degree of substitution between money and

commonly uses term-structure models to study interdependencies across maturities. Greenwood and Vayanos (2014), for instance, argue that an increase in the government's supply of long-term bonds would raise the spread between long and short rates. Oftentimes a positive correlation of yield curves for Treasury securities is taken as a sign of substitution. To identify such correlations, various papers have relied on changes in the supply of Treasury securities. Krishnamurthy and Vissing-Jørgensen (2012) and D' Aamico et al. (2012), for example, analyze how changes in the supply of T-bills affect yield spreads of different securities, suggesting imperfect substitutability between short and long-term maturities. In a similar vein, Carlson et al. (2016) use exogenous shifts in the supply of Treasury bills to document substitutability between public and private short-term debt. Lou et al. (2013), instead, focuses on the link between the primary and secondary market. They document that Treasury security prices in the secondary market decrease significantly in the few days leading up to Treasury auctions and recover shortly thereafter. This is true not only for the government security that is about to be issued, but also for related maturities.[5] Krishnamurthy and Vissing-Jørgensen (2011) study the impact of Quantitative Easing on interest rates and argue that dependencies of prices (yields) across different asset classes is key when evaluating the policy - a point which we further build upon in this paper.

To explain why bills might be complementary in the primary market we introduce a formal model. On the one hand, those with direct access to the auction might keep some of the bills they win to use them as collateral in other financial markets or to fulfill their regulatory requirements.[6] On the other hand, they can sell them in the secondary market. The quantity they demand in the primary market is, therefore, to a large extent driven by the expectations about future demand for the bills after the auction. In the secondary market, different clients demand different maturities. To avoid having to turn any of them down, to borrow the bills in the repurchase market or buy them at higher prices after the auction, bidders want to purchase enough bills of different maturities at auction. This generates complementarities across maturities in the primary market even if bills are substitutes in the secondary market. Our findings can be viewed as complementary to the existing research: Our focus is on the primary market and how to determine the supply of various maturities there. Different securities can be complementary for the primary dealers while behaving as substitutes in the aggregate.

The remainder of the article is structured as follows: Section 3.2 describes the institutional environment and the data set. Section 3.3 presents our evidence for interdependencies across maturities. Section 3.3.1 begins by documenting some patterns in the raw data that point towards interdependencies; Section 3.3.2 gives a preview of how we identify interdependen-

near-moneys" (p. 441).

[5]According to their hypothesis, "primary dealers hedge the risk they are expected to acquire at auctions by short selling similar securities, thus exerting downward price pressure in the secondary markets before these auctions" (p. 16).

[6]Participants in the Canadian Derivatives Clearing Corporation, for instance, have minimum requirements to post treasuries as collateral.

cies and summarizes the key identifying assumptions; Sections 3.3.3 and 3.3.4 describe the structural model and our estimation strategy. Estimation findings are presented in Section 3.3.5. Section 3.4 concludes. All proofs are in Appendix C.1.

3.2 Institutional Environment and Data

3.2.1 Institutional Environment

In Canada, Treasury bills are issued with three maturities: 3, 6 and 12 months. Since 1998 they are sold every second Thursday by the Bank of Canada (BoC) in three separate pay-as-bid auctions which run in parallel. All three securities have a face value of 1 million (Canadian) dollars which must be paid back at the time the security matures. There are two groups of bidders: "dealers" and "customers." Dealers are either primary dealers or government securities distributors. Customers are institutions that choose not to become primary dealers, but are large enough that the Bank of Canada requires to track their holdings separately. They can only submit bids through primary dealers. They choose not to register as dealers, perhaps to sidestep the necessary additional monitoring or to avoid the obligations dealers must fulfill to maintain their dealer status.[7] One example is Desjardin Securities. As the securities division of one of the largest Canadian financial institutions with over C$258.4 billion total assets in 2017,[8] it is a primary dealer in the bond market, but only a customer in the Treasury market. Similarly, both Casgrain & Company and JPMorgan are not registered as primary dealers and yet are very important players in the Canadian government securities markets (Hortaçsu and Kastl (2012), p. 2514).

From the time the tender call opens until the multi-unit auction closes, bidders may submit and update their bids. There are two types of bids: competitive and non-competitive. A competitive bid is a step-function with at most 7 steps. "These bids must be stated in multiples of a $1,000, subject to the condition that each individual bid be for a minimum of $100,000. Each bid shall state the yield to maturity to three decimal places" (Bank of Canada (2016), p. 2). For the most part for this paper we convert yields into prices, using

$$yield = \left(\frac{face\ value - price}{price}\right)\left(\frac{365}{days}\right) \tag{3.1}$$

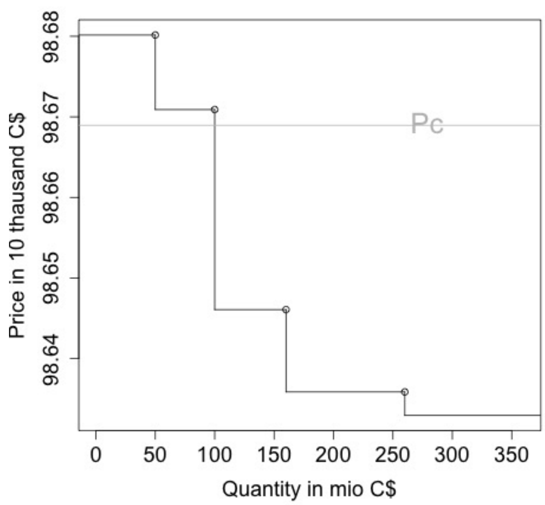
with a $face\ value$ of $C1 million and *days* denoting the days left to maturity. Using prices instead of yields makes bidding as well as demand schedules decreasing rather than increasing. The bid step-function specifies how much a bidder offers to pay for specific amounts of the asset for sale. Figure 3.1a depicts an example - the choice of the median dealer in a 12-month auction.[9] He offers to pay 98.68 thousand dollars for the first 50 million units.

[7]For more details see Sections 10 and 11 in Bank of Canada (2016).

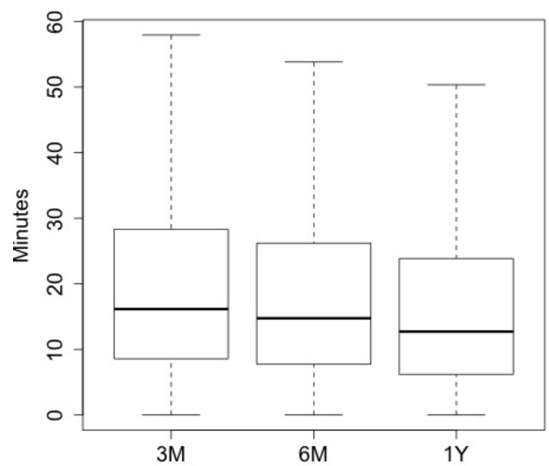
[8]See `https://www.desjardins.com/ca/about-us/desjardins/who-we-are/quick-facts/index.jsp`

[9]The median step-function is computed as follows: Determine the median number of steps in all competitive bid functions submitted by dealers, and then take the median over all (price, quantity) tuples corresponding to each step that were submitted by a dealer who submitted the median number of steps.

Figure 3.1: Bids



(a) Comp. Bid for 12M (Median Dealer)



(b) Time to Deadline
(very early & late bids excluded)

For the next 50 he offers to pay less, and so on. In addition to a competitive bid, each bidder may submit one non-competitive tender. This is a quantity order, which the bidder will win for sure, but for which he needs to pay the average price of all accepted competitive bid prices. It is capped at 3 million dollars for dealers and 5 million dollars for customers, and hence it is trivial relative to the competitive order sizes - with one exception: the Bank of Canada itself. It utilizes non-competitive bids to reduce the previously announced total amount for sale.[10] When the auction closes, the final bids are aggregated and the market clears where aggregate demand meets total supply. Everyone wins the amount they asked for at the clearing price (subject to pro-rata rationing on-the-margin in case of excess demand at the market clearing price) and pays according to what he bid. The payments for the small non-competitive tenders are computed as weighted average of accepted competitive bids. In the example of Figure 3.1a, the bidder wins roughly 100 million dollars and pays about 98, 68 thousand dollars per 1 million of face value for the first 50 units and a little less than 98.67 for the next 50.

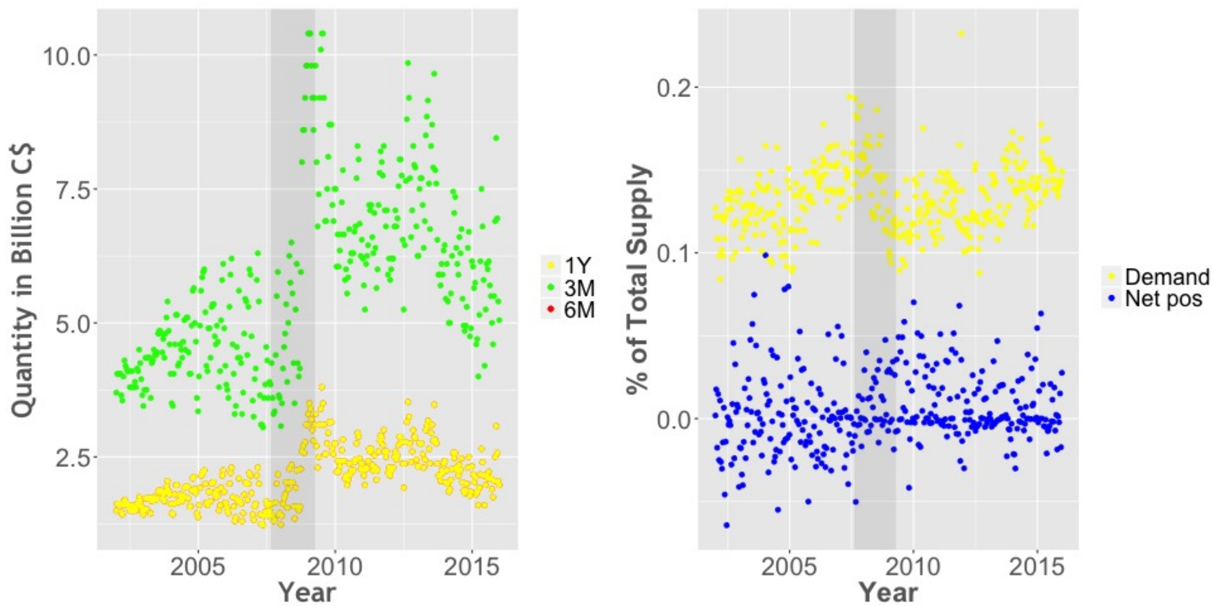
3.2.2 Data

Our data set consists of all 366 Canadian T-bill auctions between 2002 and 2015. Table 3.1 summarizes the data. On average the Bank of Canada announced issuances of C$6.41 billion for 3-month bills and C$2.47 billion for the 6 and 12-month bills per auction, of which it actually distributed roughly C$5.76 (3M) and C$2.12 billion (6/12M). The total amount issued per year was C$81 billion for the 3-month bills and C$29 billion of the longer maturities.

[10]The amounts purchased are typically divided across maturities as a proportion of what is supplied. The amounts purchased depend on the Bank's projection of expected future demand for notes and the amount of T-bills maturing over the following weeks. See Statement of Policy, 2015

Figure 3.2: Time Series of Total Supply and Individual Demand

The gray bar marks the crisis period from 2007-08-09 until 2009-04-02.



(a) Distributed Total Supply

(b) Avg Indiv. Dem. & Net position of 12M bill

Figure 3.2a displays the time series of distributed supply per maturity. Since the distributed amounts of the two longer maturities are nearly identical, the red dots corresponding to the 6-month bills are virtually perfectly masked by the yellow ones for the 12-month bills. The gray bar highlights the period of the financial crisis, from 9 August 2007 (when BNP Paribas froze three of its U.S. funds) until 2 April 2009 (when the G20 countries announced the stimulus package), during which the BoC increased its supply.

We can identify each bidder individually through a bidder ID and we know if the institution is a dealer or a customer. In total we observe 21 dealers and 76 customers over the sample period. The average auction has about 11-12 dealers and 5-6 customers participating. Roughly 71% of participants bid for all three maturities. Such "global participation" is even more regular among dealers. To keep their bidder status as government security distributor or primary dealer they have to be active in the primary market.[11] Consequently, almost all who are active in a given auction week go to all three auctions (95%).

We observe all bids submitted from the opening of the tender call until the auction closes. The entire updating period lasts one week, although, as shown in Figure 3.1b, most bids come in within 10 to 20 minutes prior to auction close. Figure 3.1b depicts box plots of the

[11] "At every auction, a primary dealer's bids, and bids from its customers, must total a minimum of 50 per cent of its auction limit and/or 50 per cent of its formula calculation, rounded upward to the nearest percentage point, whichever is less. [. . .] Each government securities distributor must submit at least one winning competitive or non-competitive bid on its own behalf or on behalf of customers, every six months." (Bank of Canada (2016), p. 12).

Table 3.1: Data Summary of 3M/6M/12M Auctions

The sample starts January 2002 and ends December 2015. There are 366 auctions per maturity. The total number of competitive bids (including updates) in the 3,6,12-month auctions is 66382, 48927, and 56721, respectively. These individual steps make up 18272, 15514 and 17077 different step-functions. The total number of non-competitive bids across maturities is 2477, 2378, and 1932. From the raw data we drop competitive bids with missing bid price (133) and competitive or non-competitive tenders with missing quantities (69). Global participation is the probability of attending the remaining auctions, conditional on bidding for one maturity. Dollar amounts are in billions of C$.

	Mean			SD			Min			Max		
	3M	6M	12M	3M	6M	12M	3M	6M	12M	3M	6M	12M
Issued amount	5.76	2.12	2.12	1.68	0.52	0.52	3.05	1.22	1.22	10.40	3.80	3.80
Dealers	11.88	11.79	11.03	0.90	0.93	0.83	9	9	9	13	13	12
Global participation	93.67	93.84	98.84	24.34	24.04	10.67	0	0	0	100	100	100
Customers	6.26	5.68	5.35	2.69	2.94	2.54	1	0	0	14	13	15
Global participation	35.66	40.13	39.46	47.90	49.02	48.88	0	0	0	100	100	100
Comp demand as %												
Of announced sup.	16.29	16.91	17.02	7.96	7.61	7.31	0.002	0.019	0.005	25	25	25
Submitted steps	4.83	4.23	4.35	1.86	1.78	1.75	1	1	1	7	7	7
Updates by dealer	2.89	2.18	2.48	3.58	2.87	3.18	0	0	0	31	31	42
Updates by customer	0.12	0.13	0.19	0.40	0.40	0.58	0	0	0	4	3	9
Non-comp dem. as %												
Of announced sup.	0.05	0.15	0.15	0.03	0.10	0.10	$5/10^5$	$4/10^5$	$2/10^3$	0.24	0.58	0.58

time at which bids arrive prior to the deadline, excluding very early outliers and bids that go in after auction closure. There are very few bids that arrive after the deadline (231 out of 57,650).[12] 22 of them win despite being late. We therefore keep late bids in our sample for estimation. Typically a dealer updates his bid (competitive or non-competitive) once or twice. The median number of updates is one. The higher average (2.26) is driven by outliers, with a maximum number of 42 updates. Customers are less likely to update with an average number of 0.1 (and a median of 0, i.e. no updates).

An average step-function of a competitive tender has 4.5 steps with little difference across maturities. Non-competitive tenders are rather small in size. On average bidders only demand 0.1% of the total (announced) supply via non-competitive tender, with a maximal share of 0.58%. Given their small magnitude, competitive tenders do not seem to play an important role in the decision process of regular participants. Our structural model will abstract from non-competitive bids, and focus solely on the decision of placing competitive bids. The BoC, on the other hand, demands substantial amounts in form of non-competitive bids to reduce the total supply on the day of the auction, which generages substantial uncertainty about the available supply - and our model will need to account for this. On average, it takes away 11.13%, (3M) 14.35% (6M), 14.26% (12M) with a maximum of 20.45% (3M), 41.66% (6M), 25.00% (12M) of the total supply it previously announced.

[12] Bids can show up as late in our data if a bidder manually phones the Bank of Canada to place a bid just before closing and the Bank takes some time to process it.

3.3 Interdependencies

Parallel auctions of different maturities might be interconnected on the supply and the demand side. On the supply side, the Bank of Canada might determine the total amounts for sale at each auction jointly, which leads to non-zero correlations of the sold amounts across maturities. To understand where interdependencies on the demand side may come from, it is useful to take a step back and ask what motivates financial institutions' activity in Treasury auctions. For one, they might want to keep some of the bills in their own inventory. Treasury bills serve as collateral in interbank markets and repo transactions and are popular to fulfill capital or liquidity requirements for save assets. Second, most bidders (primary dealers) are obligated to act as market makers in the secondary market. Therefore they buy securities of different maturities in order to sell them to clients on the secondary market. To avoid having to turn down clients with demand for different maturities in the days that follow the auction, dealers want to buy bundles of maturities. How much each bidder values the securities depends on the bank's own balance sheets and other factors that are internal to the institution. It is the presence of such private information that makes it complicated to measure interdependencies on the demand side. Bidders with private information (that might be correlated across maturities) have incentives to shade their bids so as to minimize the prices they will have to pay for each unit they win. To estimate how complementary or substitutable bills are, we first have to back out how much bidders are truly willing to pay. Without modeling how banks bid, it is not possible to measure interdependencies their true demand. In the following section, we provide suggestive empirical evidence. Although these results should be examined with caution, they do present some initial evidence in the raw data of interdependencies in the Treasury bill primary market.

3.3.1 Reduced-Form Empirical Evidence of Interdependencies

To begin our analysis, Table 3.2 displays correlations on the supply (3.2a) and demand side (3.2b) of Canadian Treasuries. The amount that the BoC announces to supply exhibits perfect positive correlation across maturities. In fact, over our long sample it always announces the exact same issuance amount for the 6 and 12-months bills. The amount it actually distributes on the auction day is also almost perfectly correlated, ranging between 0.99 and 1.[13] We observe a similar pattern on the demand side. The total amount financial institutions demand individually (via competitive or non-competitive tender) when the auction closes is highly positively correlated across maturities, about $0.91 - 0.92$. This pattern is

[13]Canadian policy markers perform stochastic simulations to determine a debt management strategy that is desirable in the longer horizon, e.g. 10 years. The model (publicly available at `https://github.com/bankofcanada/CDSM`) trades off risks and costs of different ways to decompose debt over the full spectrum of Canadian government securities. Part of the simulation routine is to specify ratios between maturities, for instance $1/4^{th}$ of each of the 3/6/12-month bills and $1/16^{th}$ of each of the 2/5/10/30-year bonds (see Table 2 on p. 35 in Bolder (2003)). Final issuance decisions are taken based on model simulations and judgment. "The typical practice [of the Bank of Canada] is to split the total amount purchased by the Bank, so that the Banks purchases approximate the same proportions of issuance by the government across the three maturity tranches" (Bank of Canada (2015) p. 5).

suggestive of banks having preferences for buying assets in some fixed proportion, pointing towards complementarities. Since the correlation between quantities actually won drops to $0.54 - 0.57$ (for all maturities) it seems that the banks do not always succeed in achieving this goal.

Table 3.2: Cross-Market Correlations

(a) Supply Side

	$\bar{Q}_{3M}$	$\bar{Q}_{6M}$	$\bar{Q}_{12M}$		Q_{3M}	Q_{6M}	Q_{12M}
$\bar{Q}_{3M}$	1.00			Q_{3M}	1.00		
$\bar{Q}_{6M}$	1.00	1.00		Q_{6M}	0.99	1.00	
$\bar{Q}_{12M}$	1.00	1.00	1.00	Q_{12M}	0.99	1.00	1.00

$\bar{Q}_m$ is the announced, Q_m the distributed supply for $m = 3, 6, 12M$

(b) Demand Side

	$q^D_{3M,i}$	$q^D_{6M,i}$	$q^D_{12M,i}$		$q^*_{3M,i}$	$q^*_{6M,i}$	$q^*_{12M,i}$
$q^D_{3M,i}$	1.00			$q^*_{3M,i}$	1.00		
$q^D_{6M,i}$	0.92	1.00		$q^*_{6M,i}$	0.57	1.00	
$q^D_{12M,i}$	0.91	0.91	1.00	$q^*_{12M,i}$	0.54	0.57	1.00

$q^D_{m,i}$ is bidder i's demand, $q^*_{m,i}$ the amount won for $m = 3, 6, 12M$

Another piece of evidence suggesting dependencies across auctions concerns updating behavior by dealers. Observing their customer orders, dealers may update their own bids. This can be because the customer bids provide information just about competition or also about the fundamental security value (Hortaçsu and Kastl (2012)). The demand for bills across auctions are likely interconnected if dealers, upon observing a customer order flow (which may be concentrated only in one maturity), update their own bids across all maturities. To be more concrete, say a dealer observes a customer bid in the 3-month auction. This triggers the dealer to update his own bid for the 3-month bill. If his demand for 3,6 and 12-month bills are interrelated, this should then also lead to an update of bids for the other maturities. To get a preliminary look at this pattern we run the following Probit regression on competitive bids placed by dealers:

$$update_{i,m} = \alpha_i + \sum_m I_m \left(\beta_m customer_m + \delta_{m,-m} customer_{-m}\right) + \varepsilon_{i,m}. \quad (3.2)$$

To avoid double counting, we count each step-function (as in Figure 3.1a) as one observation. The dependent variable *update* takes value 1 if the dealer updated his bid in an auction, and 0 otherwise. I_m is an indicator variable equal to 1 if the update occurs in the auction for maturity m. The independent variables $customer_l$ (for $l = m$ or $-m$) are also indicator variables. They are created in two different ways. In the more conservative specification (1) $customer_l$ takes value 1 only if the dealer received a competitive order by his customer for maturity l immediately before he takes action in auction m himself. The second specification builds on this benchmark but takes a longer sequence of events into account. It acknowledges that it takes time to calculate bids, enter them manually (which until 2019 is the rule rather than exception) and transfer them electronically. Table 3.3 provides an example of such a sequence. It shows the last 10 minutes of events of a dealer before auction closure on the 10th February of 2015. Having observed a customer in the 3-month auction, he takes action himself and places several bids in a row. Specification (1) assigns value 0 to $customer_{3M}$ in the 6-month auction because the dealer has not received an order for the 3-month maturity immediately before bidding on his own behalf for the 6-month bills (second to last column). He first bids for the 12-month bills. The second specification assigns a value of 1 (last column). Here $customer_l$ is 1 for all bids the dealer places in a sequence (each with a time difference of 20 seconds) if he has received an order for maturity l within one minute before he places his own bid in auction m, or the latest order the dealer achieved is for maturity l.

Table 3.3: Sequence of Events of a Dealer on 02/10/2015 in last 10 Min Before Closure

			Update in 12M for order of 3M		Update in 6M for order of 3M	
Bid by	Time	Maturity	(1)	(2)	(1)	(2)
Customer	10:19:52	3M	.	.	.	.
Dealer	10:21:59	1Y	1	1	0	0
Dealer	10:22:17	6M	0	0	0	1
Dealer	10:22:34	3M	0	0	0	0
Dealer	10:26:52	1Y	0	0	0	0
Dealer	10:27:16	1Y	0	0	0	0
Customer	10:28:34	3M	.	.	.	.
Dealer	10:28:44	3M	0	0	0	0

Table 3.4 displays the coefficients, estimated from specifications (1) and (2) in columns (1) and (2), respectively. The significant positive $\hat{\beta}_m$ coefficients support existing evidence by Hortaçsu and Kastl (2012) on dealer updating. They found that dealers respond to customer orders by updating their bids within the same auction. The significantly positive $\hat{\delta}_{m,-m}$ suggest that dealers update their bids in other maturities as well. As expected, the level of significance increases when taking into account the fact that dealers' bids are in practice hardly ever simultaneous, but instead placed in close sequence. Taken together, the evidence suggests cross-maturity updating by dealers.

3.3.2 A Preview of Our Identification Strategy

The exploratory reduced form regressions described above are clearly insufficient to quantify the substitution patterns satisfactorily. Our goal is to consistently estimate a parameter that measures by how much a bidder's marginal willingness to pay (MWTP) for some quantity of T-bills with maturity m changes the more he owns of the other maturities $-m$. As a first step we must understand what drives the MWTP in the primary market. Below we introduce a formal model that captures the key motives for purchasing bills in the primary market. We show that the true MWTP can be approximated by a linear function. To be more precise, let

Table 3.4: Probability of Dealer Updating Bids

		Dependent variable:	
		update	
Coefficient	Verbal description	(1)	(2)
$\hat{\beta}_{3M}$	update in $3M$ after order for $3M$	0.533***	0.711***
		(0.056)	(0.053)
$\hat{\delta}_{3M,6M}$	update in $3M$ after order for $6M$	0.405***	0.531***
		(0.064)	(0.061)
$\hat{\delta}_{3M,12M}$	update in $3M$ after order for $12M$	0.303***	0.446***
		(0.057)	(0.054)
$\hat{\delta}_{6M,3M}$	update in $6M$ after order for $3M$	0.086	0.248***
		(0.063)	(0.059)
$\hat{\beta}_{6M}$	update in $6M$ after order in $6M$	0.848***	0.929***
		(0.076)	(0.070)
$\hat{\delta}_{6M,12M}$	update in $6M$ after order in $12M$	0.729***	0.762***
		(0.080)	(0.074)
$\hat{\delta}_{12M,3M}$	update in $12M$ after order for $3M$	0.556***	0.664***
		(0.070)	(0.066)
$\hat{\delta}_{12M,6M}$	update in $12M$ after order for $6M$	0.120**	0.244***
		(0.059)	(0.056)
$\hat{\beta}_{12M}$	update in $12M$ after order for $12M$	0.828***	0.934***
		(0.061)	(0.059)
Constant		0.476***	0.448***
		(0.007)	(0.007)
Observations		39,271	39,271
Log Likelihood		−23,593.080	−23,350.990
Akaike Inf. Crit.		47,206.170	46,721.990

*p<0.1; **p<0.05; ***p<0.01

bidder i of type $s^g_{m,i,\tau}$ in bidder group $g \in \{d = \text{dealer}, c = \text{customer}\}$ at time τ during the auction week have the following willingness to pay for amount q_m in auction m conditional on winning q_{-m} of the other two maturities and keeping a share $(1-\kappa_m)$ on its own balance sheet

$$v_m(q_m, q_{-m}, s^g_{m,i,\tau}) = \alpha + (1-\kappa_m)s^g_{m,i,\tau} + \lambda_m q_m + \delta_m \cdot q_{-m}. \tag{3.3}$$

The vector of δ_m parameters measure the interdependencies across maturities. Take the example of the $m = 3M$ auction, where $q_{-m} \equiv \begin{pmatrix} q_{6M} & q_{12M} \end{pmatrix}'$ and $\delta_m \equiv \begin{pmatrix} \delta_{3M,6M} & \delta_{3M,12M} \end{pmatrix}$. If $\delta_{3M,6M} < 0$, bidders are willing to pay less for any amount of the 3-month maturity the more they purchase of the 6-month bills, hence the bills are substitutes. When $\delta_{3M,6M} > 0$ they are complementary, and independent if $\delta_{3M,6M} = 0$.

Estimating our parameters of interest consistently is challenging for two main reasons. First, the bank has private information about how much it values the securities. In our model, $s^g_{m,i,\tau}$ is the bank's private signal (or an index aggregate of a multidimensional signal). This generates incentives to misinterpret the true MWTP. Just like in the well-known first-price auction, bidders shade their bids to reduce the total payments they must make to win. By looking at the bids we are thus unable to differentiate between bidders reducing their bids for strategic reasons or because they are purchasing a substitute or complementary good at the same time (Problem 1: Bid-shading).[14] Secondly, even if the bidder wanted to report his true MWTP $v_m(q_m, q_{-m}, s^g_{m,i,\tau})$, the disconnected auction design does not allow it. By the rules of the auction, a bidder can, in auction m, only submit a one-dimensional bidding step-function (such as in Figure 3.1a) that depends on amounts of security m not on securities $-m$ (Problem 2: Disconnected market design). Summarizing both challenges: We observe bidding functions that specify a price for amounts of one maturity only, q_m, not the true MWTP that is a function of all maturities $v_m(q_m, q_{-m}, s^g_{m,i,\tau})$ without knowing $s^g_{m,i,\tau}$.

Our two-stage estimation procedure solves both of these problems. First, we estimate the joint distribution of market clearing prices and recover how much each bank would bid if it were bidding truthfully. This solves the problem of strategic bid-shading. Here we extend the structural estimation techniques developed by Hortaçsu (2002), Kastl (2011) and Hortaçsu and Kastl (2012) to the case of simultaneous auctions of potentially related goods. Our estimates are consistent under the identifying assumptions that (i) private information about all maturities $s^g_{i,\tau} \equiv \begin{pmatrix} s^g_{3M,i,\tau} & s^g_{6M,i,\tau} & s^g_{12M,i,\tau} \end{pmatrix}$ of each bidder i at each time τ he updates his bid is conditional on observed auction and date characteristics *iid* across bidders i and auction days, and (ii) that all bidders are ex-ante symmetric within their bidder group (dealer or customer) and play a (type-) symmetric BNE each time new bills are issued.

[14] Generally so-called " demand-reduction" can be a sever problem in multi-unit auctions in which bidders have demand for more than one unit (e.g. Ausubel et al. (2014)). In our case, however, bid-shading should play a minor role. This is because Treasury bills are highly liquid in secondary market trading. Conditional on observables, such as the when-issued price of these bills, or the spot price in the secondary market, bidders can infer one another's preferences fairly accurately.

Given the disconnected market design, the bidding schedule a bidder would submit if it were truthful, call it $\tilde{v}_m(q_m, s^g_{m,i,\tau})$, is not his true MWTP $v_m(q_m, q_{-m}, s^g_{m,i,\tau})$. This is because his actual marginal benefit from winning amount q_m depends on how much he will win of the other assets: $q^*_{-m,i}$. Since auctions take place in parallel, he does not know how much he will win. In equilibrium, these random quantities $\boldsymbol{q^*_{-m,i}}$ need to be integrated out:

$$\tilde{v}_m(q_m, s^g_{m,i,\tau}) = \mathbb{E}[v_m(q_m, \boldsymbol{q^*_{-m,i}}, s^g_{m,i,\tau})| \text{ win } q_m].$$

In the first stage of our estimation procedure we estimate $\tilde{v}_m(q_m, s^g_{m,i,\tau})$. In addition, we estimate the joint distribution of market clearing prices which allows us to estimate the conditional expectation $\mathbb{E}[\boldsymbol{q^*_{-m,i}}| \text{ win } q_m]$. Given the true MWTP is linear as assumed in (3.3) we can then estimate the parameters of interest, δ_m, in a linear regression with bidder-auction-time fixed effects that control for $\alpha + (1 - \kappa_m)s^g_{m,i,\tau}$.

We now proceed to describing the model and estimation strategy, before presenting our estimation results. Throughout, random variables will be denoted in **bold**.

3.3.3 The Model

M perfectly divisible goods, indexed m are auctioned in M separate pay-as-bid auctions, run in parallel. In each auction, there are two groups (g) of bidders: dealers (d) and customers (c). We assume that the total number of potential dealers N_d and customers N_c is commonly known, and denote the total number of bidders by $N = N_c + N_d$. Over the course of the auction, new information may arrive at a discrete number of time slots $\tau = 0, ..., \Gamma$. How much each bidder bids each τ depends on how much he is actually willing to pay. Before setting up the auction game we introduce a simple model that captures the key driving factors of individual demand in the primary market.

Micro-Foundation of Individual Demand

For simplicity in this section, we restrict the number of maturities to $M = 2$. Generalizing our micro-foundation to more than two maturities is straightforward but mathematically cumbersome and brings no major, additional insights.

As highlighted above, financial institutions participate in Treasury auctions for different purposes. They have private information about how much they need the bills. Formally, we let a bidder i of group g draw a private signal at the time τ he places his bid: $s^g_{i,\tau} \equiv \left(\boldsymbol{s^g_{1,i,\tau}} \ldots \boldsymbol{s^g_{M,i,\tau}}\right)$. This type might be multi-dimensional. To account for differences between bidder groups, it may be drawn from different distributions for customers and dealers.

Assumption 2. *Dealers' and customers' private signals $s^d_{i,\tau}$ and $s^c_{i,\tau}$ are for all bidders i independently drawn from common atomless distribution functions F^d and F^c with support $[0, 1]^M$ and strictly positive densities f^d and f^c.*

The private type determines how much a bidder benefits from keeping a share $(1-\kappa_m) \in [0,1)$ of the purchased bill m in his own inventory or to fulfill existing customer orders. Similar to the type, κ_m (as well as all other exogenous parameters of our model) could be bidder group-specific. A bidder of type $s^g_{i,\tau}$ obtains the following gross benefit from "consuming" amounts $(1-\kappa_1)q_1$ and $(1-\kappa_2)q_2$:

$$U(q_1, q_2, s^g_{i,\tau}) = s^g_{1,i,\tau}(1-\kappa_1)q_1 + s^g_{2,i,\tau}(1-\kappa_2)q_2. \tag{3.4}$$

Bidders, in particular dealers, function as market makers in the secondary market where they distribute the rest of the bills $\{\kappa_1 q_1, \kappa_2 q_2\}$ among other investors who are yet to arrive. To incorporate future resale opportunities we let there be a second stage following the primary auction. In the secondary market a (mass of) client(s) with random demand $\{\boldsymbol{x_1}, \boldsymbol{x_2}\}$ arrives to the bidder. Equivalently you may imagine that there are two types of clients, each with a random demand for one of the two maturities. For simplicity we assume that each of $\{\boldsymbol{x_1}, \boldsymbol{x_2}\}$ is on the margin uniformly distributed on $[0,1]$ but allow both amounts to be correlated. More specifically, $\{\boldsymbol{x_1}, \boldsymbol{x_2}\}$ assumes the following (cupola) density $f(x_1, x_2) = 1 + 3\rho(1-2F_1(x_1)(1-2F_2(x_2))$ with marginal distributions $F_m(x_m) = x_m$ and correlation parameter $\rho \in \left[-\frac{1}{3}, +\frac{1}{3}\right]$.[15]

The bidder sells to clients who arrive as long as he has enough of the maturities for resale: $x_m \leq \kappa_m q_m$. Selling x_m brings a payment of $p_m x_m$. The prices depend on the clients' willingness to pay, or the aggregate demand in the secondary market more generally. For simplicity we assume that it is linear and symmetric across maturities. More specifically, the inverse demand schedule for maturity 1 in the secondary market takes the following form:

$$p_1(x_1, x_2 | q_1, q_2) = \begin{cases} a - bx_1 - ex_2 & \text{for } x_1 \leq \kappa_1 q_1 \text{ and } x_2 \leq \kappa_2 q_2 \\ a - bx_1 & \text{for } x_1 \leq \kappa_1 q_1 \text{ and } x_2 > \kappa_2 q_2 \\ 0 & \text{for } x_1 > \kappa_1 q_1 \text{ and } x_2 > \kappa_2 q_2. \end{cases} \tag{3.5}$$

The price function for maturity 2 is analogous. It splits into three cases. In the first, clients for both bills arrive and the bidder has enough of both in his portfolio for resale. The bidder charges a bundle price of $\{p_1(x_1, x_2|q_1, q_2), p_2(x_1, x_2|q_1, q_2)\}$ for selling $\{x_1, x_2\}$. In the second case the bidder can only sell maturity 1. This might be because only clients with demand for this maturity arrive or because the bidder does not have enough of the other maturity in his inventory for resale, $x_2 > \kappa_2 q_2$. The price the bidder charges is independent of the maturity he does not sell, $p_1(x_1, x_2|q_1, q_2) = a - bx_1$. Finally, if the bidder does not hold enough of either bill to satisfy the demand of client(s) who arrive he cannot sell at all. Notice that the magnitudes of the resale prices are characterized by three parameters $\{a, b, e\}$. A higher intercept $a > 0$ increases the bidder's bargaining power, and with it the price he can charge for each unit he sells. Parameter $b > 0$ governs the price-sensitivity of clients. Large clients (who demand more) have more negotiating power and can drive down the price. When $e > 0$ bills are substitutes in the secondary market, and vice versa for complements.

[15] In the literature this joint distribution is known as Farie-Gumbel-Morgenstern (FGM) copula.

Selling $\{x_1, x_2\}$ generates a resale revenue of

$$revenue(x_1, x_2|q_1, q_2) = p_1(x_1, x_2|q_1, q_2)x_1 + p_2(x_1, x_2|q_1, q_2)x_2. \tag{3.6}$$

Turning down clients is costly for the bidder. An unhappy client is, for instance, less likely to contact the bidder again in the future. In reality, a bidder might even want to lend the security a client demands in the repo market so as to avoid loosing his customer in the longer run. This is costly for the bidder because it is expensive to borrow or buy additional Treasury bills from other financial institutions in the secondary market when demand is high. In our model, bidders face the following cost function:

$$cost(x_1, x_2|q_1, q_2) = \begin{cases} 0 & \text{if } x_1 \leq \kappa_1 q_1 \text{ and } x_2 \leq \kappa_2 q_2 \\ \gamma x_1 & \text{if } x_1 > \kappa_1 q_1 \text{ and } x_2 \leq \kappa_2 q_2 \\ \gamma x_2 & \text{if } x_1 \leq \kappa_1 q_1 \text{ and } x_2 > \kappa_2 q_2 \\ \gamma x_1 x_2 & \text{if } x_1 > \kappa_1 q_1 \text{ and } x_2 > \kappa_2 q_2. \end{cases} \tag{3.7}$$

The cost function captures the idea that it is more costly to turn down larger clients, i.e. those with larger demand. The important feature for our results is that it is supermodular in x_1, x_2, i.e. has increasing differences.[16] This means that the marginal cost from turning down a client who demands one maturity is higher the larger the order for the other maturity.

Taken together, a bidder expects to derive the following payoff from winning q_1, q_2 at time τ in the primary market:

$$V(q_1, q_2, s^g_{i,\tau}) = U(q_1, q_2, s^g_{i,\tau}) + \mathbb{E}\left[revenue(\boldsymbol{x_1}, \boldsymbol{x_2}|q_1, q_2) - cost(\boldsymbol{x_1}, \boldsymbol{x_2}|q_1, q_2)\right]. \tag{3.8}$$

The gross payoff determines how much a bidder is willing to pay on the margin. Consider auction 1. At time τ the bidder is willing to pay $v_1(q_1, q_2, s^g_{i,\tau}) = \frac{\partial V(q_1,q_2,s^g_{i,\tau})}{\partial q_1}$ for amount q_1 conditional on winning q_2 of the other maturity. The appendix shows that $v_1(\cdot, \cdot, s^g_{i,\tau})$ is a third-order polynomial for any $s^g_{i,\tau}$. It can be approximated by a linear function. Taking the first Tayler expansion around $(\mathbb{E}[\boldsymbol{x_1}], \mathbb{E}[\boldsymbol{x_2}]) = (1/2, 1/2)$ we obtain the following result.

Proposition 1. *The marginal willingness to pay of a bidder with type $s^g_{m,i,\tau}$ for amount q_m conditional on winning q_{-m} in the other auction can be approximated by*

$$v_m(q_m, q_{-m}, s^g_{m,i,\tau}) = \alpha + (1-\kappa_m)s^g_{m,i,\tau} + \lambda_m q_m + \delta_m q_{-m} \text{ for } m = 1, 2 - m \neq m \tag{3.3}$$

where $\alpha, \lambda_m, \delta_m$ are polynomials of the exogenous parameters $\{\kappa_1, \kappa_2, \gamma, \rho, a, b, e\}$.

The higher the private marginal benefit $s^g_{1,i,\tau}$ from keeping a share $(1-\kappa_1)$ of the bill for personal usage, the more the bidder is willing to pay. Bills might be substitutable or complementary depending on the underlying exogenous parameter.

[16] Supermodularity is for functions that map from $\mathbb{R}^n \to \mathbb{R}$ equivalent to increasing differences: $cost(x_1', x_2'|q_1, q_2) - cost(x_1, x_2'|q_1, q_2) \geq cost(x_1', x_2|q_1, q_2) - cost(x_1, x_2|q_1, q_2)$ for $x_1' \geq x_1$ and $x_2' \geq x_2$.

To build an intuition for this result, let us contrast the extreme cases where the bidder sells all of maturity 1 ($\kappa_1 = 0$), all of maturity 2 ($\kappa_2 = 0$), or keeps all of both ($\kappa_1 = \kappa_1 = 1$) and the demand of clients is stochastically independent ($\rho = 0$). Then

$$v_1(q_1, q_2, s^g_{1,i,\tau}) = \begin{cases} s^g_{1,i,\tau} & \text{if } \kappa_1 = 0 \\ \frac{1}{4}\kappa_1(b\kappa_1^2 - 2\gamma) + (1-\kappa_1)s^g_{1,i,\tau} + \kappa_1^2((a - b\kappa_1) + \frac{1}{2}\gamma)q_1 & \text{if } \kappa_2 = 0 \\ \frac{1}{8}(2(b+e) - 6\gamma) + ((a-b) - \frac{1}{4}e + \frac{7}{8}\gamma)q_1 + \frac{1}{4}(3\gamma - 2e)q_2 & \text{if } \kappa_1 = \kappa_2 = 1. \end{cases}$$

When buying only for its own account ($\kappa_1 = 0$) a bidder is willing to pay the marginal value that the bill bring his own institution $s^g_{1,i,\tau}$. When he anticipates that he will sell at least some of maturity 1 his MWTP in auction 1 decreases in q_1 as long as his clients are sufficiently price-elastic (i.e. b is sufficiently high). If he sells all of both maturities ($\kappa_1 = \kappa_2 = 1$) the MWTP is independent of his private type $s^g_{1,i,\tau}$. How much he is willing to pay for one maturity now hinges on the amount he wins of the other maturity. Whether bills are substitutes or complements in the primary market depends on how large γ is relative to e. More generally one can derive the following corollary which will be useful when interpreting our estimation results later on. It holds for the general case where clients' demand might be correlated ($\rho \neq 0$) and the bidder keeps any amounts of the bills ($\kappa_1, \kappa_2 \in [0, 1]$).

Corollary 5. *Securities in the primary market become more complementary when*
(i) they are weaker substitutes in the secondary market ($e \downarrow$),
(ii) it is more costly to turn down clients ($\gamma \uparrow$), or
(iii) it is more likely that clients with demand for different maturities arrive ($\rho \uparrow$).

The Auctions

In modeling the auction process we build on Hortaçsu and Kastl (2012)'s model of a stand-alone auction. Motivated by the previous section, we assume that the marginal willingness to pay is linear.

Assumption 3. *The marginal willingness to pay of a bidder with type $s^g_{m,i,\tau}$ for amount q_m conditional on purchasing q_{-m} of the other two securities $-m$ is*

$$v_m(q_m, q_{-m}, s^g_{m,i,\tau}) = \alpha + (1-\kappa_m)s^g_{m,i,\tau} + \lambda_m q_m + \delta_m \cdot q_{-m}. \tag{3.3}$$

with $\lambda_m < 0$, $|\delta_m| < \lambda_m$ and α sufficiently high such that the marginal willingness to pay does not drop below 0 for any amount that might be for sale.

A bid in auction m consists of a set of quantities in combination with prices. It is a step-function which characterizes the price the bidder would like to pay for each amount.

Assumption 4. *In auction m each bidder has the following action set each time he places an offer:*

$$A_m = \begin{cases} (b_m, q_m, K_m) : dim\,(b_m) = dim(q_m) = K_m \in \{1, ..., \overline{K}_m\} \\ b_{m,k} \in [0, \infty) \;\; and \;\; q_{m,k} \in [0,1] \\ b_{m,k} > b_{m,k+1} \;\; and \;\; q_{m,k} > q_{m,k+1} \forall k < K_m. \end{cases}$$

Notice that $q_{m,k} \in [0,1]$. It represents the share of total supply. This allows us to compare bids in auctions with different sizes of supply. A bid of 0 denotes non-participation.

To capture the updating process of bids prior to auction closure, we assume that new information may arrive at each time slot τ. At $\tau = 0$, a bidder draws an *iid* random variable $\Psi_i \in [0,1]$. It is one dimension of the bidder's private signal and thus unobservable to his competitors. It corresponds to the mean of an *iid* Bernoulli random variable, Ω_i, which determines whether the bidder's later bids will make it in time to be accepted by the auctioneer. More specifically, for $\tau > 0$, the bidder's information set includes the realizations $\omega_i \in \{0,1\}$ of Ω_i, where $\omega_i = 1$ means that the bid of time τ will make it in time. This gives an incentive to bid at each arrival of new information because there might not be an opportunity to bid successfully later on due to an adverse Bernoulli draw.

At each time τ all customers who want to place an order are matched to a dealer given that the rules of the auction do not allow for customers submitting their own bids. The dealer can observe his customer's bid. This provides him with additional information at time τ - one that is unavailable to other dealers or customers. A dealer might have had the same or a different customer in all three auctions. Denoting the information obtained from observing a customers' bids at time τ in auction m by $Z_{m,i,\tau}$, dealer i's information set is $\theta^g_{i,\tau} = (s^g_{i,\tau}, Z_{1,i,\tau}, Z_{2,i,\tau}, Z_{3,i,\tau})$. If he only has a customer in one auction, say for maturity 1, $\theta^g_{i,\tau} = (s^g_{i,\tau}, Z_{1,i,\tau})$, and so on. Notice that by Assumption 2 $(\boldsymbol{s^g_{i,\tau}}, \boldsymbol{Z_{i,\tau}})$ are independent across dealers and time. However, $\boldsymbol{s^g_{i,\tau}}$ and $\boldsymbol{Z_{i,\tau}}$ can be correlated within a dealer across τ, for example as further customer orders arrive later in addition to the earlier ones.

Definition 4. *A pure-strategy is a mapping from the bidder's set of private information at each time τ to the action space of all three auctions: $\Theta^g_{i,\tau} \to A_1 \times A_2 \times A_3$.*

A choice in auction m by a bidder with information $\theta^g_{i,\tau}$ may be summarized as bidding function $b^g_{m,i,\tau}(\cdot, \theta^g_{i,\tau})$ or equivalently as demand function $y^g_{m,i,\tau}(\cdot, \theta^g_{i,\tau})$. The latter specifies how much the agent demands at each admissible price. When auction m closes at $\tau = \Gamma$, the auctioneer aggregates the individual demands of the bidders' final bids. The market clears at the lowest price P^c_m at which aggregate demand, denoted $\sum_{i=1}^{N_c} y^c_{m,i,\Gamma}(p_m, \theta^c_{i,\Gamma}) + \sum_{i=1}^{N_d} y^d_{m,i,\Gamma}(p_m, \theta^d_{i,\Gamma})$ satisfies aggregate supply. The latter is the amount for sale announced prior to the auction net of what the Bank of Canada demands in form of non-competitive bids during the auction plus all other competitive tenders by bidder i's competitors.

Assumption 5. *Supply* $\{\boldsymbol{Q_1}, \boldsymbol{Q_2}, \boldsymbol{Q_3}\}$ *is a random variable distributed on* $[\underline{Q}_1, \overline{Q}_1] \times [\underline{Q}_2, \overline{Q}_2] \times [\underline{Q}_3, \overline{Q}_3]$ *with strictly positive marginal density conditional on* $s^g_{i,\tau}$ $\forall i, g = c, d$ *and* τ.

If aggregate demand equals total supply exactly there is a unique market clearing price P^c_m. Each bidder wins how much he demanded at the market clearing price and pays for all units according to his individual price offers. When there are several prices at which total supply equals aggregate demand by all bidders, the auctioneer chooses the highest one. Finally, in the event of excess demand at the market clearing price, bidders are rationed pro-rata on-the-margin.[17]

Denoting the amounts bidder i gets allocated by $q^c_i = \begin{pmatrix} q^c_{1,i} & q^c_{2,i} & q^c_{3,i} \end{pmatrix}$ when submitting $b^g_{i,\tau}(\cdot, \theta^g_{i,\tau}) \equiv \left(b^g_{1,i,\tau}(\cdot, \theta^g_{i,\tau}) \quad b^g_{2,i,\tau}(\cdot, \theta^g_{i,\tau}) \quad b^g_{3,i,\tau}(\cdot, \theta^g_{i,\tau})\right)$ his total surplus is

$$TS(b^g_{i,\tau}(\cdot, \theta^g_{i,\tau}), s^g_{i,\tau}) = V(q^c_i, s^g_{i,\tau}) - \sum_{m=1}^{3} \int_0^{q^c_{m,i}} b^g_{m,i,\tau}(x, \theta^g_{i,\tau}) dx \tag{3.9}$$

in the event in which τ is the time of his final bid, with $V(q^c_i, s^g_{i,\tau})$ given by (3.8). It is the total utility he achieves from obtaining the amounts he wins minus the total payments he must make (the area below his bidding function up to the amount he wins). Ex-ante, when placing the bid, the bidder neither knows how much he will win nor at which price the market will clear. His optimal choice maximizes the total surplus he expects to gain.

Definition 5. *A BNE is a collection of functions* $b^g_{i,\tau}(\cdot, \theta^g_{i,\tau})$ *that for each bidder* i *and almost every information* $\theta^g_{i,\tau}$ *at each time* τ *maximizes the expected total surplus,* $\mathbb{E}[TS(b^g_{i,\tau}(\cdot, \theta^g_{i,\tau}), s^g_{i,\tau})]$.

We will focus on type-symmetric BNE of the auction game in which bidders who are ex-ante identical play the same strategies. Dealers who draw the same type play the same function, and similarly for customers:

$$b^d_{i,\tau}(\cdot, \theta^d_{i,\tau}) = b^d(\cdot, \theta^d_{i,\tau}) \text{ and } b^c_{i,\tau}(\cdot, \theta^c_{i,\tau}) = b^c(\cdot, \theta^g_{i,\tau}) \ \forall i, \tau.$$

Across bidder groups strategies might be asymmetric.

[17] "Under this rule, all bids above the market clearing price are given priority, and only after all such bids are satisfied, the remaining marginal demands at exactly price $P^c = p$ are reduced proportionally by the rationing coefficient so that their sum exactly equals the remaining supply. An alternative rationing rule would, for example, not give bids at higher prices priority." (Kastl (2011), p. 980-981). The rationing coefficient satisfies $R_m(P^c_m) = \frac{Q_m - TD^+_m(P^m_c)}{TD_m(P^c_m) - TD^+_m + (P^c_m)}$ where $TD_m(P^c_m)$ denotes the total demand at price P^c_m, and $TD^+_m(P^c_m) = \lim_{p_m \downarrow P^c_m} TD_m(p_m)$.

3.3.4 Estimation Strategy

First Stage of the Estimation Strategy

To solve the problem of strategic bid-shading we recover what the bidder would bid if he were truthful in the first stage of our estimation strategy by extending Hortaçsu (2002), Kastl (2011, 2012) and Hortaçsu and Kastl (2012) to the case of simultaneous auctions of potentially related goods. To find out which marginal valuations rationalize the observed bids we must first characterize the optimality conditions for the type-symmetric BNE of the game.

Bidding incentives in simultaneous pay-as-bid auctions are similar to those in an isolated auction (see Wittwer (2018b)). To fix ideas, we begin the discussion with the benchmark case of auctions of independent goods. Securities in our model are unrelated if all δ parameters are equal to 0. In this case gross utility is additively separable across maturities and the willingness to pay for one maturity $v_m(q_m, s^g_{m,i,\tau})$ is independent of the amount allocated to this bidder in auctions of other maturities. In addition, both markets clear separately. A bid price offered for good 1 will not affect the payment the agent has to make for good 2 because the agent's demand for good 1 can, by the rules of a standard pay-as-bid auction, only depend on the price for good 1 and not the price of good 2. Since neither utility nor payments are interrelated, strategic incentives are identical to those in an isolated auction m. In determining his best reply to all others, the bidder can, therefore, focus on each auction in isolation. If the bidder knew the residual supply curve when choosing his bids, he would just pick a point on this curve that maximizes his total surplus. Yet, when making his choices, he does not know this curve as it depends on the random total supply and the private information of his competitors. He thus has to integrate out the uncertainty about the market clearing price and evaluate marginal benefits and costs of changing a bid. The marginal cost is losing the surplus on the last infinitesimal unit demanded, which happens exactly when price is between bids defined by the k^{th} and $k+1^{st}$ step. The marginal benefit is saving the difference between these bids whenever the market clearing price ends up being actually weakly lower than b_{k+1}.

Proposition 2 (Independent Goods). *Consider a bidder i of group g with private information $\theta^g_{i,\tau}$ who submits $\hat{K}_m(\theta^g_{i,\tau})$ steps in auction m at time τ. Under Assumptions 2-5 in any type-symmetric BNE every step k in his bid function $b^g_m(\cdot, \theta^g_{i,\tau})$ has to satisfy*

$$v_m(q_{m,k}, s^g_{m,i,\tau}) = b_{m,k} + \frac{\Pr\left(b_{m,k+1} \geq \boldsymbol{P^c_m} | \theta^g_{i,\tau}\right)}{\Pr\left(b_{m,k} > \boldsymbol{P^c_m} > b_{m,k+1} | \theta^g_{i,\tau}\right)} (b_{m,k} - b_{m,k+1}) \ \forall k < \hat{K}_m(\theta^g_{i,\tau})$$

and $b_{m,k} = v_m(\bar{q}_m(\theta^g_{i,\tau}), s^g_{m,i,\tau})$ at $k = \hat{K}_m(\theta^g_{i,\tau})$ where $\bar{q}_m(\theta^g_{i,\tau})$ is the maximal amount the bidder may be allocated in the equilibrium.

This equation allows us to estimate the marginal valuations on the left-hand-side that rationalize the bids we observe for all steps $b_{m,k}$ of all bidders at all times and auctions. Following the idea in Hortaçsu (2002) the approach is to estimate the distribution of the market clearing price using a resampling procedure. It relies on the assumption that private information is not interdependent across bidders, so that "each bidder i cares about others bidding strategies only insofar as they affect the distribution of bidder i's residual supply" (Hortaçsu and McAdams (2010), p. 842). The choice of bid by bidder i transforms the distribution of the residual supply into the distribution of the market clearing price. In the easiest set-up of a standard multi-unit auction with no updating of bids in which N potential, ex-ante symmetric bidders who draw independent private information and play the symmetric BNE in T auctions with identical covariates, the resampling procedure works as follows: Fix bidder i. For all bidders that did not bid in an auction, augment the data with their bids being 0. Draw a random subsample of $N-1$ bid vector with replacement from the sample of NT bids in the data set. Construct bidder i's realized residual supply were others to submit these bids to determine the realized market clearing price. Repeating this routine many times gives a consistent estimate of the distribution of the market clearing price. Our setup is, even if there are no interdependencies across auctions, more complicated than this benchmark. First, we have two bidder groups (dealers and customers) which may be ex-ante asymmetric. Second, bidders may update their bids within an auction with dealers observing their customers' bids. Hortaçsu and Kastl (2012) have extended the resampling procedure for this more complex environment.[18] We describe how we extend their method for stand-alone auctions to the cases of parallel auctions below.

It is highly unlikely that demands for Treasury bills of different maturities are independent, in particular when the maturities are similar. Bidders take this interconnection across auctions into account when determining their optimal bidding strategies. Consider an auction for maturity $m = 1$. When preferences are no longer separable across maturities, the agent's marginal willingness to pay for amount q_1 depends on how much of the other goods he gets allocated, $v_1(q_1, q_{-1}, s^g_{1,i,\tau})$. Ideally, he would want to condition his price $b_{1,k}$ for amount $q_{1,k}$ on how much he will purchase of the other securities in equilibrium, $q^*_{-1,i} \equiv \begin{pmatrix} q^*_{2,i} & q^*_{3,i} \end{pmatrix}'$. Since the rules of the auction do not allow the participants to express their preferences in this way, they have to integrate out the uncertainty. Conditional on winning $q_{1,k}$, which happens when $b_{1,k} \geq \boldsymbol{P^c_1} > b_{1,k+1}$, a bidder expects a marginal benefit of $\mathbb{E}\left[v_1\left(q_{1,k}, \boldsymbol{q^*_{-1,i}}, s^g_{1,i,\tau}\right) \middle| b_{1,k} \geq \boldsymbol{P^c_1} > b_{1,k+1}, \theta^g_{i,\tau}\right]$. Analogous to the decision process in an isolated auction, the agent equates the benefit of winning the bid with its marginal cost. Since auctions clear separately the cost is identical to the cost in an isolated auction with one difference. With stochastic dependence across auction markets market clearing prices are connected. With M maturities, they are drawn from a joint M-dimensional distribution.

[18] Their procedure works as follows: Start by drawing N_c customer bids from the empirical distribution of customer bids. If a customer did not participate, replace his bid by a 0. For each customer bid vector, draw a corresponding dealer bid. If a zero customer bid is drawn, draw from the pool of uninformed dealers (those who did not observe any customer bids). If a nonzero customer bid is drawn, draw from the pool of dealers' bids, which have been submitted having observed a "similar" customer bid with equal probabilities. Those are customer bids whose quantity-weighted bid price are sufficiently close (according to a pre-defined bandwidth). The resulting estimate of the distribution of clearing prices is consistent (Hortaçsu and Kastl (2012)).

Proposition 3 (Related goods)**.** *Consider a bidder i of group g with private information $\theta^g_{i,\tau}$ who submits $\hat{K}_m(\theta^g_{i,\tau})$ steps in auction m at time τ. Under Assumptions 2-5 in any type-symmetric BNE every step k in his bid function $b^g_m(\cdot, \theta^g_{i,\tau})$ has to satisfy*

$$\tilde{v}_m(q_{m,k}, s^g_{m,i,\tau}|\theta^g_{i,\tau}) = b_{m,k} + \frac{\Pr\left(b_{m,k+1} \geq \boldsymbol{P^c_m}|\theta^g_{i,\tau}\right)}{\Pr\left(b_{m,k} > \boldsymbol{P^c_m} > b_{m,k+1}|\theta^g_{i,\tau}\right)}(b_{m,k} - b_{m,k+1}) \; \forall k < \hat{K}_m(\theta^g_{i,\tau})$$

with

$$\tilde{v}_m(q_{m,k}, s^g_{m,i,\tau}|\theta^g_{i,\tau}) \equiv \mathbb{E}\left[v_m\left(q_{m,k}, \boldsymbol{q^*_{-m,i}}, s^g_{m,i,\tau}\right)\middle| b_{m,k} \geq \boldsymbol{P^c_m} > b_{m,k+1}\theta^g_{i,\tau}\right]$$

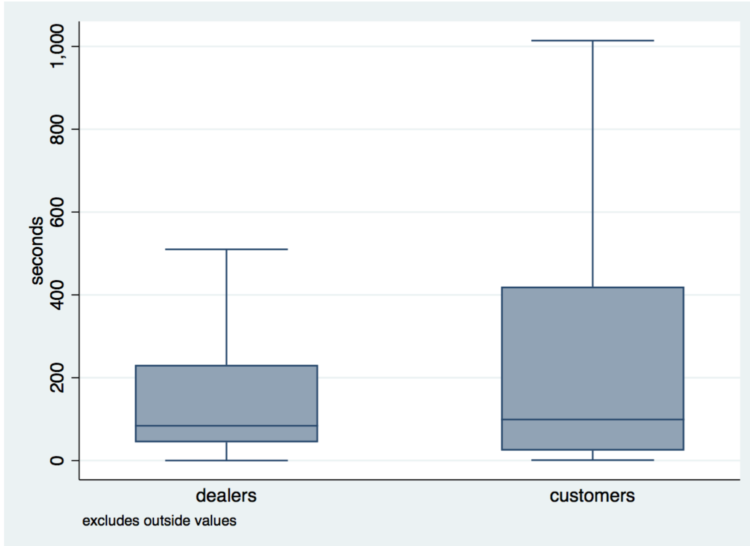
for $m = 1 \dots M$ with $-m \neq m$, and $b_{m,k} = \tilde{v}_m(\bar{q}_m(\theta^g_{i,\tau}), s^g_{m,i,\tau}|\theta^g_{i,\tau})$ at $k = \hat{K}_m(\theta^g_{i,\tau})$ where $\bar{q}_m(\theta^g_{i,\tau})$ is the maximal amount the bidder may be allocated in an equilibrium.

Analogous to the stand-alone auction, we can estimate the marginal valuations (LHS) by estimating the distribution of residual supply curves, now jointly for all maturities. With $M = 3$ parallel auctions, the benchmark resampling procedure of Hortaçsu (2002) must be changed in that a choice of a bidder is now a triplet of bidding functions submitted on a given auction day. Fixing such a triplet of bids submitted by a bidder, one then draws a random subsample of $N-1$ bid vector triplets with replacement from the sample of NT bids in the data set, and constructs bidder i's realized residual supply $\forall m$ were others to submit these bids to determine the realized clearing prices $P^c = \left(P^c_{3M} \quad P^c_{6M} \quad P^c_{12M}\right)$, and the amount i would have won $q^*_i = \left(q^*_{3M,i}, q^*_{6M,i}, q^*_{12M,i}\right)$ for all q^*_i, P^c. Repeating this procedure a large number of times provides an estimate of the joint distribution of market clearing prices and, equally important, the corresponding amount of each security i would win.

Our more complex environment with dealers and customers who may update their bids requires a more complicated procedure (see Hortaçsu and Kastl (2012)). There are two complications when auctions are not considered separately. First, bids in different auctions are not submitted at the exact same time given electronic or human delays (see the example in Table 3.3 above). In our procedure, we define bids to be "simultaneous" if they are the closest bids of all bids a bidder places within 200 seconds, or they are the last bids he makes before the auction deadline, i.e. his final bids. Setting an upper bound of 200 seconds seems sensible when looking at the number of seconds between bids across maturities which we know were determined "simultaneously". Those are cases where the bidder does not update his bids over the course of the auctions. On average 551/383 seconds pass between such bids for different maturities by dealers/customers. Excluding outliers, the time reduces as shown in the box plots of Figure 3.3.

Second, a customer might place his order via different dealers in an auction week. He might, for instance, go via one dealer in the 3-month auction and via another in the 6-month auction. Furthermore, two bids for the same maturity but by different customers might go through the same dealer. Neither of these two cases happens more than a handful of times. Therefore, we assume that the information set of dealers who observe the same customer is independent across maturities, conditional on his own signal. In addition, we restrict the

Figure 3.3: Time Between Bids of Those Who Do Not Update

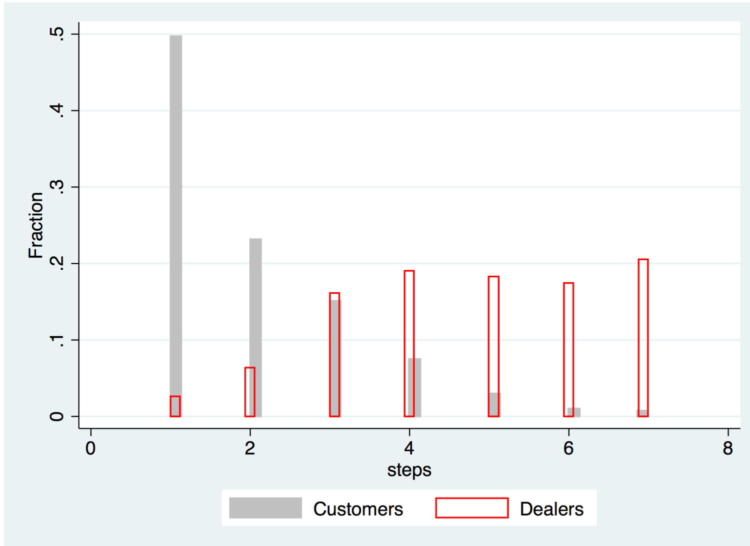


number of possible observed customer bids to two. Given that most customers only submit one bid and that there are many more dealers than customers in a typical auction, this simplifying restriction should not be problematic.

With these simplifications our procedure goes as follows: Start by drawing N_c customer bids from the empirical distribution of customer bids at date t. If a customer did not participate in one auction, replace his bid by 0. For each customer, find the dealer(s) who observed this customer's bid(s). If the customer submitted only one bid, take the dealer who observed it. If the customer submitted more than one bid, draw uniformly over dealer-bids having observed this customer. Finally, if the total number of dealers drawn is at this point lower than the total number of potential dealers, draw the remaining bids from the pool of uninformed dealers, i.e. those who do not observe a customer bid in any of the three auctions.

In our main specification we impose marginal valuations $\tilde{v}_m(\cdot, s^g_{m,i,\tau}|\theta^g_{i,\tau})$ to be weakly decreasing. Increasing valuations would imply increasing equilibrium bidding functions, which cannot be submitted by the rules of the auction. Furthermore, to correct for outliers that occur occasionally due to small values of the denominator in the estimated (marginal) hazard rate of the market clearing price, $\hat{\Pr}\left(b_{m,k} > P^c_m > b_{m,k+1}|\theta^g_{i,\tau}\right)$, we trim our estimated marginal values. More specifically, we restrict each one of them to be lower than the bidder's maximal bid plus a markup of about 0.5 basis points (C\$50 for 1Y, C\$25 for 6M, C\$12.5 for 3M). Our results are robust with regards to the restriction on valuations, and not sensitive to the choice of this markup as long as it is not too high (see Appendix C.2 for more details).

Figure 3.4: Steps by Bidder Groups



Second Stage of the Estimation Strategy

Our resampling procedure delivers a consistent estimate of the joint distribution of market clearing prices and the amount bidder i wins in equilibrium conditional on the information he has at time τ. This allows us to estimate how much he expects to win of the other maturities if he were to win a given quantity in maturity m.

$$\hat{q}^*_{t,-m,i,\tau,k} = \mathbb{E}\left[\boldsymbol{q^*_{t,-m,i}} | b_{t,m,i,\tau,k} \geq \boldsymbol{P^c_{t,m}} > b_{t,m,i,\tau,k+1}, \theta_{t,i,\tau}\right] + \varepsilon^q_{t,m,i,\tau,k} \tag{3.10}$$

Moreover, using Proposition 3, we can use the marginals of this joint distribution to obtain an estimate of how much bidder i would be willing to pay at step k at time τ in auction m of week t given the observed bid.

$$\hat{v}_{t,m,i,\tau,k} = \mathbb{E}\left[v_m\left(q_{t,m,i,\tau,k}, \boldsymbol{q^*_{t,-m,i}}, s^g_{m,i,\tau}\right) \middle| b_{t,m,i,\tau,k} \geq \boldsymbol{P^c_{t,m}} > b_{t,m,i,\tau,k+1}, \theta_{t,i,\tau}\right] + \varepsilon^v_{t,m,i,\tau,k} \tag{3.11}$$

Assuming linearity of the marginal willingness to pay (Assumption 3), we can now estimate the following linear regression with auction-bidder-time fixed effects $u_{t,m,i,\tau} \equiv \alpha + (1 - \kappa_m)s^g_{t,m,i,\tau}$

$$\hat{v}_{t,m,i,\tau,k} = u_{t,m,i,\tau} + \lambda_m q_{t,m,i,\tau,k} + \delta_m \cdot \hat{q}^*_{t,-m,i,\tau,k} + \varepsilon_{t,m,i,\tau,k} \tag{3.12}$$

for $m = 1 \ldots M, m \neq m$ on a subsample with competitive bids of more than one step to identify the parameters of interest. Figure 3.4 shows that it is the case for virtually all dealer bids: almost all submit more than one step.

3.3.5 Estimation Results

Our data covers the time period from 2002 until 2015. It includes the years of the financial crisis, during which one may worry that participants' preferences may have been different than in normal times. In what follows we first discuss estimation results based on the full time period. Thereafter we split the sample into the years prior to the crisis (2002-2006), the financial crisis (2007-2010) and the years that followed (2011-2015) to analyze whether preferences change across time.

In our main specification we restrict attention to dealers. This is only in order to avoid potential issues arising from possible selection on the customers' side due to a relatively small sample of customer bids with two or more steps. We also restrict our sample to bids that arrive less than 30 minutes to auction closure. This covers the vast majority of bids (recall Figure 3.1b, p. 35) and avoids possible confounds due to unobserved heterogeneity (e.g., public information arriving over time). It excludes really early bids and occasional instances where dealers updated their bids many times. Typically they update once or twice. While our theory allows for many updates, in order to simplify our resampling algorithm we restrict the number of possible observed customer bids to two. In Appendix C.2 we display several other specifications that use bids submitted by dealers and customers and different time restrictions. In addition we run all regressions (3.12) using $b_{t,m,i,\tau,k}$ to complement our results using $\hat{v}_{t,m,i,\tau,k}$. If bidders are truthful (at every step), the bids they place should be a good approximation for their marginal valuations conditional on that step being marginal. Using the submitted price bids rather than our estimated marginal valuations as independent variable therefore has two purposes. First, it provides a sanity check of our estimated marginal values. Estimates should not differ tremendously in size. Second, it gives an idea of whether bid-shading ($b_{t,m,i,\tau,k} - \hat{v}_{t,m,i,\tau,k}$) is sensitive to interdependencies across auctions.

Full Time Period

Estimation results of regression (3.12) are displayed in Tables 3.5 - 3.7. In line with our model, we report all findings with bids and marginal valuations expressed in C$ (prices) not in basis points (of yields). Whenever we quote a number in basis points it is the estimated value from the corresponding regressions performed with yields (not prices).[19] All quantities are expressed in C$10 million (the 10×face value). The first column uses our estimated valuations as independent variable, the third the bids we observe.

Before discussing the estimated degree of interdependency (the δ parameters), consider first by how much a dealers' MWTP for maturity m changes in q_m (the λ parameters of the first column in all tables). This helps provide a sense of the parameter magnitudes. As expected, marginal utility is strictly decreasing (all λ's are significantly negative). They are not large in magnitude, however, indicating that valuations are fairly flat with respect to quantity. Increasing the amount of the 3-month bills by 10 million, for example, decreases a dealers' marginal benefit from owning the 3-month bill by C$0.744 or about 0.03 basis points. Given

[19] As a rule of thumb, 100/50/25 C$ of a 12M/6M/3M bill are approximately 1 basis point of a yield.

Table 3.5: 3M Auction (Full Time Period)

	Estimated Valuation		Observed Bid	
λ_{3M}	-0.744^{***}	(-165.83)	-0.747^{***}	(-171.51)
$\delta_{3M,6M}$	$+0.334^{***}$	(6.80)	$+0.393^{***}$	(8.25)
$\delta_{3M,1Y}$	$+0.355^{***}$	(6.69)	-0.366^{***}	(7.11)
Constant	995533.2^{***}	(2574206.36)	995528.8^{***}	(2652525.62)
Observations	45994		45994	

t statistics in parentheses, * $p < 0.05$, ** $p < 0.01$, *** $p < 0.001$

Table 3.6: 6M Auction (Full Time Period)

	Estimated Valuation		Observed Bid	
λ_{6M}	-3.095^{***}	(-132.97)	-3.060^{***}	(-135.55)
$\delta_{6M,3M}$	$+0.0051$	(0.13)	$+0.119^{**}$	(3.14)
$\delta_{6M,1Y}$	$+0.0934$	(0.84)	$+0.399^{***}$	(3.69)
Constant	991509.1^{***}	(1494437.31)	991500.2^{***}	(1540913.01)
Observations	34267		34267	

t statistics in parentheses, * $p < 0.05$, ** $p < 0.01$, *** $p < 0.001$

Table 3.7: 12M Auction (Full Time Period)

	Estimated Valuation		Observed Bid	
λ_{1Y}	-6.476^{***}	(-154.51)	-6.086^{***}	(-154.61)
$\delta_{1Y,3M}$	-0.629^{***}	(-8.51)	-0.261^{***}	(-3.76)
$\delta_{1Y,6M}$	$+0.846^{***}$	(4.34)	$+1.614^{***}$	(8.82)
Constant	981316.6^{***}	(841131.53)	981280.2^{***}	(895599.08)
Observations	42830		42830	

t statistics in parentheses, * $p < 0.05$, ** $p < 0.01$, *** $p < 0.001$

the large average allocated amounts (about C$544 million of 3M bills per dealer), however, there is a more sizable difference between the value of the first and last Treasury bill: about 1.49 basis points. The other two maturities exhibit similar patterns: In the 6-month bill auction, the bidder's marginal valuation drops by C$3.095. This translates to a difference in values for the first and last won Treasury bill (on average C$204 million per dealer) of about 1.32 basis points. For one year T-bills, the difference is about 1.38 basis points. Estimates are similar when using the observed bids as approximation of the true valuation (third column). Given the small price variation in the submitted bid step-functions (recall Figure 3.1a) this finding should not be that surprising. Intuitively, financial institutions have a rather precise idea of the price at which the primary market will clear since all securities we consider are highly liquid. To avoid paying unnecessarily high prices, they submit bids that vary closely around the clearing price they expect conditional on publicly available information (such as current yield curves in the when-issued or secondary market).

With these "own" effects in mind, we can turn to the discussion of our main parameters of interest. We find significant interdependencies in the 3-month and 12-month T-bill auctions. Relative to the λ parameters, these interdependencies are not negligible in size. Consider first the 3-month bills (first column of Table 3.5). In the 3-month auction both δ parameters are positive, suggesting that the 3-month bill is complementary to the longer maturities. Estimates are similar when using bids as independent variables (third column). When a bidder expects to win 10 million more of the 6-month (12-month) bill, his true value in the 3-month auction increases by C$0.334 (C$0.355), which is a bit less than half of the amount it decreases in response to winning 10 million more of the 3-month bill. This effect implies an increase in the valuation for 3-month T-bill when going from an allocation with no other T-bill of other maturity to an allocation corresponding to the observed averages of about 0.5 basis points. For bidders in the 12-month bill auction, the 6-month bills are complementary whereas the 3-month bill is a substitute (first column of Table 3.7). A similar conclusion can be drawn when using the observed bids as independent variable (third column). This is not the case for the 6-month bill auction. When using our estimated valuations we find no evidence for interdependencies over the full sample period (Table 3.6). As we will show in the next section, however, this may be driven by changes in preferences during the financial crisis. When using the observed bids as independent variables (third column), both δ parameters are significantly positive, hinting towards complementarities.

Generally, the estimates when using the estimated true valuations (first column) and those where we approximate this valuation by the submitted bid (third column) are not that similar for the longer maturities. One lesson to draw from this exercise is that bids might be directly informative about the slope of marginal valuation with respect to own quantity, but are on their own not very informative about interdependencies. For the most liquid maturity, the price bid seems to be a fairly good approximation of the true valuation. Estimates of the two specifications are very similar. For the longer maturities, however, there are more differences. These differences might be driven by higher bid-shading in the auctions for the 6 and 12-month bills.

Financial Crisis

We now split our sample into a period prior the crisis, the crisis and years that followed. Generally, banks might have unusual needs for securities in turbulent times, for instance because of they are more cautious to take on risk. For Canada Allen et al. (2011) document an increase in demand for liquidity following the demise of Lehman Brothers. Furthermore, the most recent financial crisis triggered an enforcement in stricter liquidity and capital requirements which might have caused structural change in the demand for Treasury securities.

Tables 3.8 - 3.10 report estimation results of regression (3.12) in our main specification. During the crisis years the 3-month bill was a (weak) substitute for bidders in the 6-month as well as the 12-month auction. Going from none to the observed average allocation of the 3-month bill (C$632 million per dealer) decreases the bidder's willingness to pay in the 6/12-month auction by about 0.5/1.2 basis points. The magnitude of all parameters (including the slope with respect to own quantity) is larger with the exception of $\delta_{6M,1Y}$. In the 3-month auction (Table 3.8), for example, λ_{3M} decreases from -0.677 before the crisis to -1.044 during the crisis and then increases again to -0.589. The size of the $\delta_{3M,6M}$ parameter changes from 0.278 to 0.420 to 0.374. More generally, compared to before the crisis, all δ parameters, but $\delta_{1Y,6M}$ which remained about the same, increased. After the crisis, all maturities are complementary, if interdependent. According to our theory complementarities in the primary market increase when it is more costly to turn down clients, demand for different maturities after the auction is with higher probability similar in size, or bills are weaker substitutes/stronger complements in the secondary market (Corollary 5).

Table 3.8: 3M Auction (Pre/During/Post Crisis)

	Pre		Crisis		Post	
λ_{3M}	-0.677^{***}	(-95.27)	-1.044^{***}	(-98.18)	-0.589^{***}	(-111.95)
$\delta_{3M,6M}$	$+0.218^{***}$	(3.77)	$+0.420^{***}$	(4.12)	$+0.374^{***}$	(5.16)
$\delta_{3M,1Y}$	$+0.310^{***}$	(5.27)	$+0.419^{***}$	(3.54)	$+0.328^{***}$	(4.33)
Constant	992189.8^{***}	(1920929.74)	994346.6^{***}	(1011262.51)	997787.0^{***}	(2139133.26)
Observations	10822		12530		22642	

t statistics in parentheses, * $p < 0.05$, ** $p < 0.01$, *** $p < 0.001$

Table 3.9: 6M Auction (Pre/During/Post Crisis)

	Pre		Crisis		Post	
λ_{6M}	-3.113^{***}	(-80.97)	-4.328^{***}	(-70.30)	-2.529^{***}	(-103.48)
$\delta_{6M,3M}$	$+0.0257$	(0.43)	-0.374^{***}	(-3.85)	$+0.231^{***}$	(5.22)
$\delta_{6M,1Y}$	$+0.265^{*}$	(2.19)	$+0.546^{*}$	(2.11)	$+0.774^{***}$	(4.79)
Constant	985555.4^{***}	(1141446.47)	989994.8^{***}	(555714.09)	995766.8^{***}	(1297220.30)
Observations	9072		9221		15974	

t statistics in parentheses, * $p < 0.05$, ** $p < 0.01$, *** $p < 0.001$

Table 3.10: 12M Auction (Pre/During/Post Crisis)

	Pre		Crisis		Post	
λ_{1Y}	-6.834^{***}	(-100.91)	-9.482^{***}	(-84.46)	-5.184^{***}	(-109.77)
$\delta_{1Y,3M}$	-0.366^{***}	(-3.47)	-1.748^{***}	(-9.76)	$+0.0180$	(0.19)
$\delta_{1Y,6M}$	$+1.104^{***}$	(4.96)	$+3.090^{***}$	(6.77)	$+0.966^{***}$	(3.31)
Constant	969569.1^{***}	(641567.61)	977987.2^{***}	(315592.78)	990597.3^{***}	(671392.12)
Observations	12753		10247		19830	

t statistics in parentheses, * $p < 0.05$, ** $p < 0.01$, *** $p < 0.001$

3.4 Conclusion

In this paper we study interdependencies in the demand for securities of different maturities. Using data from Canadian Treasury bill auctions over a 15 year period, we find that 3, 6 and 12-month bills are oftentimes complementary in the primary market for Treasury bills. To explain our findings we present a model that captures the interplay between the primary and secondary market. We argue that the typical bidder of a primary auction buys bills not only for his own balance sheets but also (or even primarily) to distribute them in the secondary market where different clients demand different maturities. He anticipates that it will be costly to turn down clients in case he did not buy sufficiently many bills at auction, or to satisfy their demand by purchasing the bills from other financial institutions at higher prices. This generates complementarities in the primary market even if T-bills are substitutes in other financial markets.

In future work we plan to derive and discuss policy recommendations for the auctioneers. Since demand interdependencies are likely to arise in many (if not most) applications of multi-unit auctions due to various reasons (such as budget constraints), our goal is to demonstrate that taking them seriously and identifying them explicitly can help auctioneers in achieving their objectives. In the context of auctions of government debt, this objective is to raise revenue (sell debt) at lowest possible cost (i.e., lowest interest rate). More specifically, we plan to use our estimates to conduct a back-of-the-envelope counterfactual exercise to quantify how much the government could have saved (in terms revenue extracted) if the Bank of Canada had decided on the split of T-bills across the three maturities more strategically.

APPENDIX

Appendix A

Chapter 1

A.1 Proof of Theorem 1

Theorem 1 is equivalent to Theorem 2 when types are drawn from a degenerated distribution. To prove Theorem 1 it suffices to make the following changes in the proof of Theorem 2: First, replace the equilibrium winning quantity $\boldsymbol{q_i^*}$, its distribution and density by the per-capita supply $\boldsymbol{q^*}$, its distribution and density. Second, notice that the hazard rate of $\boldsymbol{q^*}$ is weakly decreasing by the assumption that total supply is drawn from a distribution with decreasing hazard rate. Similarly, the second condition for equilibrium existence of Theorem 2 is always satisfied without private types because $\frac{\partial x(b(q))}{\partial p} = \left(\frac{\partial b(q)}{\partial q}\right)^{-1}$. I invite who would like to see a full proof for the environment with symmetrically informed bidder, to consult Pycia and Woodward (2017)'s proof of Theorem 3 (pp. 46-49).

Theorem 1 in relation to the existing literature (Pycia and Woodward (2017), Holmberg (2006, 2009)):

Pycia and Woodward (2017) derive the following bid-representation

$$b^*(q) = \int_{Nq}^{\overline{Q}} v\left(\frac{y}{N}\right) d\mathcal{F}^{Nq,N}(y) \quad \text{with} \quad \mathcal{F}^{Q,N}(y) \equiv 1 - \left[\frac{1 - F_Q(y)}{1 - F_Q(Q)}\right]^{\frac{N-1}{N}} \text{ for } Q < \overline{Q}. \quad \text{(A.1)}$$

In what follows I show that our bidding functions coincide by re-formulating mine

$$b^*(q) = v(q) - \int_q^{\overline{q}} \left[\frac{1 - F_{q^*}(x)}{1 - F_{q^*}(q)}\right]^{\frac{N-1}{N}} (-1) \left(\frac{\partial v(x)}{\partial q}\right) dx \quad \text{on } [0, \overline{q}^*] \quad (1.1)$$

to match theirs. First integrate the $(-1)\cdot$ integral of function (1.1) by parts:

$$\int_q^{\overline{q}} \left[\frac{1 - F_{q^*}(x)}{1 - F_{q^*}(q)}\right]^{\frac{N-1}{N}} v'(x)dx = -v(q) + \int_q^{\overline{q}} \left[\frac{N-1}{N}\right] \left[\frac{f_{q^*}(x)}{1 - F_{q^*}(q)}\right] \left[\frac{1 - F_{q^*}(x)}{1 - F_{q^*}(q)}\right]^{-\left[\frac{1}{N}\right]} v(x)dx$$

to obtain

$$b^*(q) = \int_q^{\overline{q}} \left[\frac{N-1}{N}\right] \left[\frac{f_{q^*}(x)}{1 - F_{q^*}(q)}\right] \left[\frac{1 - F_{q^*}(x)}{1 - F_{q^*}(q)}\right]^{-\left[\frac{1}{N}\right]} v(x)dx.$$

Now change the variable of integration to $y = xN$ and use the distribution function of the total supply, $F_Q(y) = F_Q(Nx) = F_{q^*}(x)$ and $f_Q(Nx) = \frac{1}{N} f_{q^*}(x)$:

$$b^*(q) = \int_{Nq}^{\overline{Q}} \left[\frac{N-1}{N}\right] \left[\frac{f_Q(y)}{1 - F_Q(Nq)}\right] \left[\frac{1 - F_Q(y)}{1 - F_Q(Nq)}\right]^{-\left[\frac{1}{N}\right]} v\left(\frac{y}{N}\right) dy.$$

Note that this is the analogue to Holmberg (2006, 2009)'s bidding function in a pay-as-bid procurement auction (see Holmberg (2006) equation (6) on page 8 with $\bar{p} = 0$).

Finally use $\frac{\partial}{\partial y}\left\{1 - \left[\frac{1-F_Q(y)}{1-F_Q(Nq)}\right]^{\left[\frac{N-1}{N}\right]}\right\} = \left[\frac{N-1}{N}\right]\left[\frac{f_Q(y)}{1-F_Q(Nq)}\right]\left[\frac{1-F_Q(y)}{1-F_Q(Nq)}\right]^{-\left[\frac{1}{N}\right]}$ and pull $v\left(\frac{y}{N}\right)$ forward

$$b^*(q) = \int_{Nq}^{\overline{Q}} v\left(\frac{y}{N}\right) \frac{\partial}{\partial y}\left\{1 - \left[\frac{1 - F_Q(y)}{1 - F_Q(Nq)}\right]^{\left[\frac{N-1}{N}\right]}\right\} dy.$$

To obtain Pycia and Woodward (2017)'s representation (A.1), it suffices to use their auxiliary distribution function $\mathcal{F}^{Q,N}(y)$.

A.2 Proof of Theorem 2

Throughout the proof I work with the distribution of i's clearing price quantity $\boldsymbol{q_i^c}$. This is the quantity the agent wins at market clearing. When all choose the equilibrium quantities it coincides with the distribution of i's equilibrium winning quantity $\boldsymbol{q_i^*}$, but not otherwise.

Definition 6. *Define the probability that bidder i obtains at most quantity q when submitting $b_i(\cdot, t_i)$ such that $b_i(q, t_i) = p$ as*

$$G_i(p, q) \equiv \Pr\left(\boldsymbol{Q} - \sum_{j \neq i} x(p, \boldsymbol{t_j}) \leq q\right). \tag{A.2}$$

Denote its support by $[0, \overline{q}_i^c]$ and the corresponding density by $g_i(p, q)$.

Notice that the lower bound of the support is 0 because total supply may realize at $\overline{Q} = 0$. The upper bound is endogenous. Since total supply and types are bounded, however, there is a maximal amount that the bidder can win even when submitting a price of 0.

The goal is to show that there exists a pure-strategy BNE in which bidders submit

$$b^*(q, t_i) = v(q, t_i) - \int_q^{\overline{q}_i^*} \left[\frac{1 - F_{q_i^*}(x)}{1 - F_{q_i^*}(q)}\right]^{\frac{N-1}{N}} (-1) \left(\frac{\partial v(q, t_i)}{\partial q}\right) dx \text{ on } q \in [0, \overline{q}_i^*] \tag{1.4}$$

and $b^*(q, t_i) = v(q, t_i)$ on $q \in (\overline{q}_i^*, \infty)$ given that

1. the distributions of total supply and types are such that the amount an agent wins in the symmetric equilibrium $\boldsymbol{q_i^*}$ is drawn from a distribution $F_{q_i^*}$ with weakly decreasing hazard rate and strictly positive density on support $[0, \overline{q}_i^*]$

2. and the corresponding equilibrium demand schedule $b^*(\cdot, t_i)^{-1}$ is additively separable in the type on the range of relevant quantities $q \in [0, \overline{q}_i^*]$.

The second condition restricts the class of functions that I can look for: I want to find an equilibrium of the following form

$$b^*(q, t_i) = \begin{cases} b(q, t_i) & \text{for } q \in [0, \overline{q}_i^c] \\ b_T(q, t_i) & \text{for } q \in (\overline{q}_i^c, \infty) \end{cases} \tag{A.3}$$

with

$$b(q, t_i) = y^{-1}(q - \eta(t_i)) \tag{A.4}$$

where $y(\cdot)$ and $b_T(\cdot, t_i)$ are twice differentiable and strictly decreasing and $\overline{q}_i^c$ is the maximal amount the agent can win at market clearing (Definition 6). Assuming (A.4) guarantees that demand schedules are additively separable in type. To see this, solve $b(q, t_i) = p$ with $q = x(p, t_i)$ to obtain

$$x(p, t_i) = \eta(t_i) + y(p). \tag{A.5}$$

A.2.1 The Core of the Argument

The proof is long and mathematically tedious so that it helps to lay out the main line of argument before carrying it out in all details: Take the perspective of bidder i, fix his type t_i, and let all others $j \neq i$ choose as in equilibrium. My candidate equilibrium function (A.3) splits into two parts. Consider the second part first: $b_T(q, t_i)$ for $q \in (\overline{q}_i^c, \infty)$. Quantities higher than the maximal amount i can win at market clearing are unachievable. The agent never wins nor pays higher amounts than $\overline{q}_i^c$. They are irrelevant. It therefore does not matter which prices the agents bids for these amounts, provided the bidding function is differentiable and decreasing on the whole domain $\mathbb{R}^+$. In the equilibrium of the theorem I consider best replies in which the agent just behaves truthfully for irrelevant amounts:

$$b_T(q, t_i) = v(q, t_i) \text{ for } q \in (\overline{q}_i^c, \infty) = (\overline{q}_i^*, \infty) \text{ in the equilibrium.} \tag{A.6}$$

Now consider the first part: $b(q, t_i)$ for $q \in [0, \overline{q}_i^c]$. The core of the proof is to show that function (1.4) is played in the symmetric equilibrium on the domain of relevant quantities $[0, \overline{q}_i^c]$ which in equilibrium become $[0, \overline{q}_i^*]$. I must show that it is optimal for agent i to choose (1.4) in responds to all others playing it. He chooses his best reply so as to maximize his expected total surplus subject to two constraints, so-called end-point or boundary conditions. The lower bound may be some arbitrary finite price $\overline{p}$. The upper bound is given by the bidder's true marginal willingness to pay, as explained above.

$$\max_{b_i(\cdot, t_i)} \mathcal{V}(b_i(\cdot, t_i)) \text{ subject to } b_i(0, t_i) = \overline{p} < \infty \text{ and } b_i(\overline{q}_i^c, t_i) = v(\overline{q}_i^c, t_i) \tag{M}$$

where

$$\mathcal{V}(b_i(\cdot, t_i)) = \int_0^{\overline{q}_i^c} \left[\int_0^q v(x, t_i) - b_i(x, t_i) dx \right] g_i(q, b_i(q, t_i)) dq.$$

First, I derive a necessary condition for equilibria with strictly decreasing and differentiable bidding functions (Lemma 3). Since the agent must choose the same strategy as all others I can solve for the solution explicitly: When imposing symmetry across agents $[0, \overline{q}_i^c]$ of (A.3) becomes $[0, \overline{q}_i^*]$ and the necessary condition a linear differential equation (Lemma 4). The bidding function (1.4) of the Theorem is its unique solution (Lemma 5). Three auxiliary lemma then show that this candidate is indeed a BNE (Lemma 6). Auxiliary Lemma 1 verifies that function (1.4) is strictly decreasing and differentiable, Auxiliary Lemma 2 that the function fulfills the sufficient conditions of a local maximum and Auxiliary Lemma 3 that it is globally maximizing the bidder's objective functional.

A.2.2 Full Proof

Throughout the proof I will often drop the superscript * and treat all functions as functions of quantity only since the type is fixed. I denote the first and second derivatives w.r.t. to quantity (or price respectively) by $'$ and $''$, for instance: $b'(q,t_i) = \frac{\partial b(q,t_i)}{\partial q}$.

Lemma 3. *Consider a set of strictly decreasing and differentiable functions $\{b_i(\cdot, t_i)\}_{i=1}^N$. If they constitutes a BNE it must be that*

$$0 = -\left(\frac{\partial G_i(q, b_i(q,t_i))}{\partial b_i(q,t_i)}\right)[v(q,t_i) - b_i(q,t_i)] - [1 - G_i(q, b_i(q,t_i))] \tag{E}$$

is satisfied point-wise for all $q \in [0, \overline{q}_i^c]$ and all $i = 1...N$.

Proof. In order to derive necessary conditions, I first re-state the objective functional so that it depends on the distribution function of i's clearing price quantity instead of its density. Integrating by parts I obtain

$$\mathcal{V}(b_i(\cdot,t_i)) = \int_0^q [v(x,t_i) - b_i(x,t_i)]dx G_i(q, b_i(q,t_i))\Big|_0^{\overline{q}_i^c} - \int_0^{\overline{q}_i^c} [v(q,t_i) - b_i(q,t_i)]\, G_i(q, b_i(q,t_i))dq.$$

Since $G_i(0, b_i(0,t_i)) = 0, G_i(\overline{q}_i^c, b_i(\overline{q}_i^c, t_i)) = 1$ for any function and type, and $[b_i(0,t_i) - v(0,t_i)] < \infty$ by the end-point condition,

$$\begin{aligned}\mathcal{V}(b_i(\cdot,t_i)) &= \int_0^{\overline{q}_i^c} [v(x,t_i) - b_i(x,t_i)]dx - \int_0^{\overline{q}_i^c} [v(q,t_i) - b_i(q,t_i)]\, G_i(q, b_i(q,t_i))dq \\ &= \int_0^{\overline{q}_i^c} [v(q,t_i) - b_i(q,t_i)]\,[1 - G_i(q, b_i(q,t_i))]dq.\end{aligned} \tag{$\mathcal{V}$}$$

To determine the necessary conditions of the following functional

$$V = \int_0^{\overline{q}_i^c} \mathcal{F}(q, b_i(q,t_i))dq \text{ s.t. } b_i(0,t_i) = \overline{p} < \infty \text{ and } b_i(\overline{q}_i^c, t_i) = v(\overline{q}_i^c, t_i) \tag{$\mathcal{V}$}$$

$$\text{where } \mathcal{F}(q, b_i(q,t_i)) \equiv [v(q,t_i) - b_i(q,t_i)]\,[1 - G_i(q, b_i(q,t_i))] \tag{$\mathcal{F}$}$$

one constructs a class of comparison functions, $b_i(q,t_i) + \varepsilon\kappa(q)$, around the extremal $b_i \equiv b_i(q,t_i)$. V's first variation (the analogue of the first derivative) must be 0 for any variation

$\kappa(q)$. The resulting necessary condition is famously known as Euler-Lagrange Equation. In this special case where $\mathcal{F}$ does not depend on $b_i'(q,t_i)$ it simplifies to $0 = \frac{\partial \mathcal{F}(q,b_i)}{\partial b_i}$. Given $\mathcal{F}$'s functional form:

$$0 = -\left(\frac{\partial G_i(q,b_i)}{\partial b_i}\right)[v(q,t_i) - b_i] - [1 - G_i(q,b_i)]. \tag{E}$$

In the standard case, in which $\mathcal{F}$ depends on the slope of $b_i(\cdot,t_i)$, the Euler-Equation involves two arbitrary coefficients. To find them, one uses the boundary conditions through which the function must run. Here, where $\mathcal{F}$ is independent of the functions' slope, $0 = \frac{\partial \mathcal{F}(q,b_i)}{\partial b_i}$ must pass through the end points $(0,\overline{p})$ and $(\overline{q}_i^c, v(\overline{q}_i^c,t_i))$ (Elsgolc (1961) p. 31). □

Lemma 4. *Consider a strictly decreasing and differentiable function $b(\cdot,t_i)$. If it constitutes a symmetric BNE then*

$$[v(q,t_i) - b(q,t_i)] = -\left[\frac{N}{N-1}\right]\left[\frac{1 - F_{q_i^*}(q)}{f_{q_i^*}(q)}\right] b'(q,t_i) \text{ for } q \in [0,\overline{q}_i^*]. \tag{N}$$

Proof. To evaluate the Euler-Lagrange Equation at the guessed, symmetric solution (A.4), recall that the equilibrium demand function $x(\cdot,t_j)$ is of the following form

$$x(p,t_j) = \eta(t_j) + y(p). \tag{A.5}$$

Given that all others choose such strategy, agent i's necessary condition must be satisfied if he himself also plays this strategy, i.e.

$$b_i(q,t_i) = y^{-1}(q - \eta(t_i)). \tag{A.4}$$

Imposing this symmetry across agents enables me to simplify the necessary condition. The trick is to re-state the condition using the distribution of i's winning quantity in the symmetric equilibrium $F_{q_i^*}$, instead of i's clearing price quantity. The later determines the probability that i wins at most quantity q when offering *some* price for this amount. The second is this probability when choosing the *equilibrium* price.

To do so I first calculate the amount i wins when playing the equilibrium guess (A.5):

$$\begin{aligned}
\boldsymbol{Q} &= \boldsymbol{q_i^*} + \sum_{j\neq i} x(\boldsymbol{p^c},\boldsymbol{t_j}) \text{ with } \boldsymbol{p^c} = b_i(\boldsymbol{q_i^*},t_i) && \text{by market clearing}\\
\boldsymbol{q_i^*} &= \boldsymbol{Q} - \sum_{j\neq i}\eta(\boldsymbol{t_j}) - \sum_{j\neq i} y(y^{-1}(\boldsymbol{q_i^*} - \eta(t_i))) && \text{by (A.4), (A.5)}\\
\Rightarrow \boldsymbol{q_i^*} &= \left[\frac{1}{N}\right]\left\{\boldsymbol{Q} - \sum_{j\neq i}\eta(\boldsymbol{t_j}) + (N-1)\eta(t_i)\right\}. && \text{(A.7)}
\end{aligned}$$

Definition 7. *The probability that bidder t_i wins at most quantity $q \in [0,\overline{q}_i^*]$ in the symmetric equilibrium is*

$$F_{q_i^*}(q) = \Pr\left(\boldsymbol{q_i^*} \leq q\right) \;\; with\; \boldsymbol{q_i^*} \equiv \left[\frac{1}{N}\right] \left\{\boldsymbol{Q} - \sum_{j \neq i} \eta(\boldsymbol{t_j}) + (N-1)\eta(t_i)\right\} \tag{A.7}$$

Denote the corresponding density by $f_{q_i^*}(q)$*.*

To replace G_i by $F_{q_i^*}$, recall the definition of i's clearing price quantity:

$$G_i(q,p) \overset{(A.2),(A.5)}{=} \Pr\left(\boldsymbol{Q} - \sum_{j \neq i} \eta(\boldsymbol{t_j}) \leq q + (N-1)y(p)\right) \text{ for any } (q,p). \tag{A.15}$$

At $p = b(q, t_i) = y^{-1}(q - \eta(t_i))$ the distributions of the clearing price and winning quantity coincide:

$$G_i(q, b(q,t_i)) = \Pr\left(\boldsymbol{q_i^*} \leq q\right) = F_{q_i^*}(q). \tag{A.8}$$

To determine the partial derivative of $G_i(q, b_i(q, t_i))$ w.r.t. to price $b_i = b_i(q, t_i)$, I insert the guessed equilibrium function, apply the chain rule and change the random variable from $\left[\boldsymbol{Q} - \sum_{j \neq i} \eta(\boldsymbol{t_j})\right]$ to $\boldsymbol{q_i^*}$ to obtain

$$\frac{\partial G_i(q, b(q,t_i))}{\partial b_i} = \left[\frac{N-1}{N}\right] f_{q_i^*}(q)\, y'(b(q,t_i)). \tag{A.9}$$

By the definition of an inverse and the chain rule $y'(p) = \left(\frac{1}{b'(q,t_i)}\right)$, so that

$$\frac{\partial G_i(q, b(q,t_i))}{\partial b_i} = \left[\frac{N-1}{N}\right] f_{q_i^*}(q) \left(\frac{1}{b'(q,t_i)}\right). \tag{A.10}$$

With (A.8) and (A.10) I can evaluate the Euler-Lagrange equation at the guessed symmetric solution where $\bar{q}_i^c = \bar{q}_i^*$. (E) becomes

$$0 = -\left[\frac{N-1}{N}\right] f_{q_i^*}(q) \left(\frac{1}{b'(q,t_i)}\right) [v(q,t_i) - b(q,t_i)] - [1 - F_{q_i^*}(q)] \quad \text{for } q \in [0, \bar{q}_i^*].$$

Since $f_{q_i^*}(\cdot)$ is strictly positive on its support by the assumption that all densities are strictly positive on their respective support, I can rearrange this equation to obtain (N'). □

Lemma 5. $\exists!$ *function that fulfills the necessary conditions for a symmetric BNE:*

$$b(q,t_i) = v(q,t_i) - \int_q^{\bar{q}_i^*} \left[\frac{1 - F_{q_i^*}(x)}{1 - F_{q_i^*}(q)}\right]^{\left[\frac{N-1}{N}\right]} (-1)v'(x,t_i)dx \;\; \textit{for } q \in [0, \bar{q}_i^*]. \tag{1.4}$$

[0] By assumptions all $j \neq i$ play the guessed equilibrium $x(p, t_j) = \eta(t_i) + y(p)$. By definition of an inverse, $x(b(q,t_i) = y(b(q,t_i)) + \eta(t_j) = q$. By the chain rule $\left(\frac{\partial y(b(q,t_i))}{\partial p}\right) b'(q,t_i) = 1$, so that $\left(\frac{\partial y(b(q,t_i)}{\partial p}\right) = b'(q,t_i)^{-1}$. Differentiating $x(b(q,t_i) = y(b(q,t_i)) + \eta(t_j) = q$ a second time gives $\left(\frac{\partial^2 y(b(q,t_i)}{\partial p^2}\right) b'(q,t_i)^2 + \left(\frac{\partial y(b(q,t_i)}{\partial p}\right)\left(\frac{\partial^2 b(q,t_i)}{\partial^2 q}\right) = 0$. Inserting the last equation to obtain $\left(\frac{\partial^2 y(b(q,t_i)}{\partial p^2}\right) = -\left(\frac{\partial^2 b(q,t_i)}{\partial^2 q}\right) b'(q,t_i)^{-3}$.

Proof. The solution to differential equation (N') for $q \in [0, \overline{q}_i^*)$ is

$$b(q,t_i) = [F_{q_i^*}(q)-1]^{-\left[\frac{N-1}{N}\right]}\left[C + \left[\frac{N-1}{N}\right]\int_{q_0}^{q}[F_{q_i^*}(x)-1]^{-\left[\frac{1}{N}\right]}f_{q_i^*}(x)v(x,t_i)dx\right] \text{ with } q_0 \in [0,\overline{q}_i^*).$$

C is chosen to ensure that the solution passes through $(\overline{q}_i^*, v(\overline{q}_i^*, t_i))$. Here I will guess

$$C = -\left[\frac{N-1}{N}\right]\int_{q_0}^{\overline{q}_i^*}[F_{q_i^*}(x)-1]^{-\left[\frac{1}{N}\right]}f_{q_i^*}(x)v(x,t_i)dx \qquad (C)$$

and verify at the end that the resulting solution indeed goes through the upper end-point condition. Inserting (C) into the bidding function and simplifying gives the following unique solution

$$b(q,t_i) = [F_{q_i^*}(q)-1]^{-\left[\frac{N-1}{N}\right]}\left[-\left[\frac{N-1}{N}\right]\int_{q}^{\overline{q}_i^*}[F_{q_i^*}(x)-1]^{-\left[\frac{1}{N}\right]}f_{q_i^*}(x)v(x,t_i)dx\right].$$

To simplify the bidding function I integrate its integral by parts and use $F_{q_i^*}(\overline{q}_i^*) = 1$.

$$\frac{N-1}{-N}\int_{q}^{\overline{q}_i^*}[F_{q_i^*}(x)-1]^{\frac{-1}{N}}f_{q_i^*}(x)v(x,t_i)dx = [F_{q_i^*}(q)-1]^{\frac{N-1}{N}}v(q,t_i) + \int_{q}^{\overline{q}_i^*}[F_{q_i^*}(x)-1]^{\frac{N-1}{N}}v'(x,t_i)dx$$

The bidding function becomes

$$b(q,t_i) = [F_{q_i^*}(q)-1]^{-\left[\frac{N-1}{N}\right]}\left[[F_{q_i^*}(q)-1]^{\left[\frac{N-1}{N}\right]}v(q,t_i) + \int_{q}^{\overline{q}_i^*}[F_{q_i^*}(x)-1]^{\left[\frac{N-1}{N}\right]}v'(x,t_i)dx\right].$$

It simplifies to the bidding function of the theorem

$$b(q,t_i) = v(q,t_i) - \int_{q}^{\overline{q}_i^*}\left[\frac{1-F_{q_i^*}(x)}{1-F_{q_i^*}(q)}\right]^{\left[\frac{N-1}{N}\right]}(-1)v'(x,t_i)dx \text{ for } q \in [0,\overline{q}_i^*]. \qquad (1.4)$$

It remains to show that the function goes through $(\overline{q}_i^*, v(\overline{q}_i^*, t_i))$. For this I must show that

$$\lim_{q\to\overline{q}_i^*}\int_{q}^{\overline{q}_i^*}\left[\frac{1-F_{q_i^*}(x)}{1-F_{q_i^*}(q)}\right]^{\left[\frac{N-1}{N}\right]}v'(x,t_i)dx = 0.$$

To do so, separate

$$\lim_{q\to\overline{q}_i^*}\int_{q}^{\overline{q}_i^*}\left[\frac{1-F_{q_i^*}(x)}{1-F_{q_i^*}(q)}\right]^{\frac{N-1}{N}}v'(x,t_i)dx = \lim_{q\to\overline{q}_i^*}\left[1-F_{q_i^*}(q)\right]^{\frac{N-1}{-N}}\lim_{q\to\overline{q}_i^*}\int_{q}^{\overline{q}_i^*}\left[1-F_{q_i^*}(x)\right]^{\frac{N-1}{N}}v'(x,t_i)dx.$$

Applying Hospital's rule in combination with the Fundamental Theorem of calculus:

$$= -\lim_{q\to\overline{q}_i^*}\left[\frac{N-1}{N}f_{q_i^*}(q)^{\frac{-1}{N}}\right]^{-1}\lim_{q\to\overline{q}_i^*}\left[1-F_{q_i^*}(x)\right]^{\frac{N-1}{N}}v'(x,t_i)\Big|_{q}^{\overline{q}_i^*}$$

The first limit is different from zero since $f_{q_i^*}(q) > 0$ on the full support. The second limit is 0 because $F_{q_i^*}(\overline{q}_i^*) = 1$ and $v(\cdot, t_i)$ is bounded by 0. Putting both parts together, this shows that the limit is 0. □

Lemma 6. *Function (1.4) is a BNE.*

Proof. Function (1.4) is a BNE if (*i*) it is strictly decreasing and differentiable and (*ii*) agent i has no profitable deviation from submitting another function than the one that fulfills the necessary conditions of his maximization problem. The proof splits into three auxiliary lemma.

Auxiliary Lemma 1. *Function (1.4) is strictly decreasing and twice differentiable.*

Proof. Twice differentiability of $b(\cdot, t_i)$ follows immediately from the assumptions that all distribution functions and $v(\cdot, t_i)$ are twice differentiable. That $b(\cdot, t_i)$ is strictly decreasing in q can easily be verified by taking the derivative:

$$\frac{\partial b(q,t_i)}{\partial q} < 0 \quad \text{for } q \in [0, \overline{q}_i^*) \;\text{ iff }\; \int_q^{\overline{q}_i^c} \left[1 - F_{q_i^*}(x)\right]^{\frac{N-1}{N}} \left(\frac{\partial v(x,t_i)}{\partial q}\right) dx < 0.$$

This always holds given $1 - F_{q_i^*}(x) > 0$ and $\left(\frac{\partial v(x,t_i)}{\partial q}\right) < 0$ on $x \in [0, \overline{q}_i^*)$ by assumption. At the boundary point where $b(\overline{q}_i^*, t_i) = v(\overline{q}_i^*, t_i)$, the function is strictly decreasing since the marginal valuation is strictly decreasing in quantity by assumption. □

Auxiliary Lemma 2. *Function (1.4) fulfills the sufficient conditions of a local maximum.*

Proof. Sufficient conditions in variational calculus problems are generally tricky. One needs to show that the second variation (the analogue to the second derivative when maximizing w.r.t. a variable) of functional $\mathcal{V}$ defined on page 78 is positive for all values of variation $\kappa(q)$ over the interval $0 \leq q \leq \overline{q}_i^c$.[1] Here, where $\mathcal{F}$ is independent of $b_i'(q,t_i)$ the second variation is

$$\delta V_2 = \int_0^{\overline{q}_i^c} \kappa(q)^2 \left[\frac{\partial^2 \mathcal{F}(q, b_i(q,t_i))}{\partial^2 b_i}\right] dq.$$

In what follows I show that $\frac{\partial^2 \mathcal{F}(q,b_i)}{\partial^2 b_i}$ evaluated at the solution $b_i(q,t_i) = b(q,t_i)$ is at all relevant points negative:

$$\frac{\partial^2 \mathcal{F}(q, b(q,t_i))}{\partial^2 b_i} < 0 \text{ for all } q \in [0, \overline{q}_i^*]. \tag{S}$$

For this I must first specify the functional form of $\frac{\partial^2 \mathcal{F}}{\partial^2 b_i}$. From above we already know

$$\frac{\partial \mathcal{F}(q, b_i(q,t_i))}{\partial b_i} = -\left(\frac{\partial G_i(q, b_i(q,t_i))}{\partial b_i}\right)[v(q,t_i) - b_i(q,t_i)] - [1 - G_i(q, b_i(q,t_i))].$$

[1] In the standard case, in which $F \equiv F(q, b_i(q,t_i), b_i'(q,t_i))$ depends on the slope of the function one maximizes over, b_i', the second variation is known to be (Kamien and Schwartz (1993), p. 42)

$$\delta V_2 = \int_0^{\overline{q}_i^c} \left\{\kappa^2 \frac{\partial^2 F}{\partial^2 b_i} + 2\kappa\kappa' \left(\frac{\partial^2 F}{\partial b_i \partial b_i'}\right) + (\kappa')^2 \left[\frac{\partial^2 F}{\partial^2 b_i'}\right]\right\} dq.$$

Taking the partial derivative w.r.t. price $b_i(q,t_i) = b_i$ gives

$$\frac{\partial^2 \mathcal{F}(q, b_i(q,t_i))}{\partial^2 b_i} = 2\left(\frac{\partial G_i(q, b_i(q,t_i))}{\partial b_i}\right) - \left(\frac{\partial^2 G_i(q, b_i(q,t_i))}{\partial^2 b_i}\right)[v(q,t_i) - b_i(q,t_i)] .$$

Analogous to above the sufficient condition simplifies when re-formulating it using the distribution of i's equilibrium winning quantity $\boldsymbol{q_i^*}$: Transform the random variable from $\boldsymbol{Q} - \sum_{j\neq i}\eta(\boldsymbol{t_j})$ to $\boldsymbol{q_i^*}$, and recall $G_i(q,p) = \Pr\left(\boldsymbol{Q} - \sum_{j\neq i} x(p, \boldsymbol{t_j}) \leq q\right) = \Pr\left(\boldsymbol{Q} - \sum_{j\neq i}\eta(\boldsymbol{t_j}) \leq q + (N-1)y(p)\right)$ to obtain

$$\frac{\partial G_i(q, b(q,t_i))}{\partial b_i} = \left[\frac{N-1}{N}\right] f_{q_i^*}(q)\, y'(b(q,t_i)) \tag{A.9}$$

and

$$\frac{\partial^2 G_i(q, b(q,t_i))}{\partial^2 b_i} = \frac{1}{N^2}\left(\frac{\partial f_{q_i^*}(q)}{\partial q}\right)[N-1]y'(b(q,t_i))^2 + \frac{1}{N} f_{q^i}(q)\,[N-1]y''(b(q,t_i)). \tag{A.11}$$

By definition of an inverse and the chain rule

$$y'(b(q,t_i)) \overset{0}{=} \left(\frac{1}{b'(q,t_i)}\right) \tag{A.12}$$

$$y''(b(q,t_i)) \overset{0}{=} -\left(\frac{b''(q,t_i)}{b'(q,t_i)^3}\right). \tag{A.13}$$

Inserting equations (A.9), (A.11), (A.12), (A.13) into the sufficient condition (S) and multiplying both sides of the equation by $\frac{N}{N-1} b'(q,t_i)^2 > 0$ gives

$$-[v(q,t_i) - b(q,t_i)]\left\{(N-1)\left[\frac{\partial f_{q_i^*}(q)}{\partial q}\right] - f_{q_i^*}(q)\left[\frac{b''(q,t_i)}{b'(q,t_i)}\right]\right\} + 2b'(q,t_i) f_{q_i^*}(q) < 0 \tag{A.14}$$

for $q \in [0, \overline{q}_i^*]$. For the boundary point, $q = \overline{q}_i^*$, we immediately see that this condition holds. The first term drops out since $v(q,t_i) = b(q,t_i)$ and the second term is strictly negative. To check whether the condition is satisfied for $q \in [0, \overline{q}_i^*)$ I use (N') to substitute out for $b'(q,t_i)$ in combination with its fist and second derivative to substitute out for $\left[\frac{b''(q,t_i)}{b'(q,t_i)}\right]$. The sufficient condition simplifies to

$$\frac{d}{dx}\ln\left[\frac{1 - F_{q_i^*}(x)}{f_{q_i^*}(x)}\right]\Bigg|_{x=q} > \left[\frac{v'(q,t_i)}{v(q,t_i) - b(q,t_i)}\right] \text{ for } q \in [0, \overline{q}_i^*). \tag{S}$$

The RHS is always negative, because $[v(q,t_i) - b(q,t_i)] > 0$ for $q \in [0, \overline{q}_i^*)$ and $v(\cdot, t_i)$ is strictly decreasing. The sufficient condition is therefore fulfilled whenever the LHS is weakly positive. This is the case when the winning quantity is drawn from a distribution with weakly decreasing hazard rate, which holds by assumption.

I conclude that the second variation is negative for all admissible functions $\kappa(\cdot)$. The critical function $b(\cdot, t_i)$ is a local maximum. □

Auxiliary Lemma 3. *Function (1.4) is a global maximum.*

Proof. In what follows I first show that there is at most one function that satisfies the necessary condition of the agent's maximization problem (Part 1). From Lemma 5 we know that there is such a function, namely function (1.4). To prove that this unique local maximum is a global maximum, I verify that it cannot be optimal to choose a function that lies on the boundaries of the function space in the final step (Part 2).

Part 1. *There is at most one function that satisfies the necessary condition for a symmetric equilibrium.*

First, following Pycia and Woodward (2017) we know that any best reply must be strictly decreasing on relevant quantities $[0, \overline{q}_i^c]$. In a sequence of lemmas they prove that bidding functions in pay-as-bid auctions in which agents are uncertain about the total supply and all share the same type are strictly decreasing (pp. 34-37). Their Lemmas 1-4 extend to my environment with private information with minor modifications: Fix a type profile $t \equiv (t_1, ...t_N)$ and consider agent i. The type-dependent valuation $v_i(\cdot, t_i)$ and bidding function $b_i(\cdot, t_i)$ replace $v_i(\cdot)$ and $b_i(\cdot)$ in Pycia and Woodward (2017). The clearing price now maps from total supply and the fixed profile of all types into the space of prices: $p(Q, t)$. The set of relevant quantities is $[0, \overline{q}_i^c]$. Its upper bound is denoted by $\varphi^i(p(\overline{Q})$ in Pycia and Woodward (2017).

It remains to show that there is at most one function that fulfills the necessary conditions within the class of strictly decreasing functions. For this recall that any such function $b_i(\cdot, t_i)$ must be such that for any fixed $q \in [0, \overline{q}_i^c]$,

$$0 = -\left(\frac{\partial G_i(q, b_i(q, t_i))}{\partial b_i(q, t_i)}\right)[v(q, t_i) - b_i(q, t_i)] - [1 - G_i(q, b_i(q, t_i))]. \tag{E}$$

In other words, (E) must hold point-wise, for all relevant quantities $q \in [0, \overline{q}_i^c]$. To show that there is at most one critical function I must show that for any fixed $q \in [0, \overline{q}_i^c]$ there is at most one bid price $b_i \equiv b_i(q, t_i)$ that makes the condition bind. Since all candidate functions are strictly decreasing in quantity, I can equivalently show that for any fixed relevant price b_i there is at most one q that guarantees (E). To show this, I first simplify the condition using the assumption that players $j \neq i$ choose $x(p, t_j) = y(p) + \eta(t_j)$. Then I fix some relevant price b_i and show that the condition is strictly increasing in q by the assumption that i's equilibrium winning quantity has a decreasing hazard rate. This implies that for any relevant b_i there is at most one q.

To simplify (E), recall that for any fixed $\{q, p\}$ and given $x(p, t_j) = y(p) + \eta(t_j)$

$$G_i(q, p) \overset{(A.5)}{=} \Pr\left(\boldsymbol{Q} - \sum_{j \neq i} \eta(\boldsymbol{t_j}) \leq q + (N-1)y(p)\right) \tag{A.15}$$

$$G_i(q, p) = F_Z\left(q + (N-1)y(p)\right) \text{ with } \boldsymbol{Z} \equiv \boldsymbol{Q} - \sum_{j \neq i} \eta(\boldsymbol{t_j}) \tag{A.16}$$

where F_Z, f_Z denote the CDF and density of auxiliary variable $\boldsymbol{Z}$. Applying the Chain Rule, condition (E) becomes

$$0 = -f_Z(q + (N-1)y(b_i))(N-1)y'(b_i)\left[v(q,t_i) - b_i\right] - \left[1 - F_Z\left(q + (N-1)y(b_i)\right)\right] \quad \text{(A.17)}$$

for any $\{q, b_i\}$ such that $z = q + (N-1)y(b_i)$ lies in the support of $\boldsymbol{Z}$. By assumption all random variables have strictly positive density on their respective supports so that $f_Z > 0$ on its support. I can therefore divide by the density. Furthermore I can divide by $(N-1)y'(b_i)$ as $y'(b_i) < 0$ and $N > 1$. The condition rearranges to

$$0 = \left[\frac{1 - F_Z\left(q + (N-1)y(b_i)\right)}{f_Z(q + (N-1)y(b_i))}\right]\left(\frac{1}{(N-1)(-1)y'(b_i)}\right) - \left[v(q,t_i) - b_i\right]. \quad (E)$$

Now, fix some relevant price b_i. By assumption i's winning equilibrium quantity, defined $\boldsymbol{q_i^*} = \left[\frac{1}{N}\right]\{\boldsymbol{Z} + (N-1)\eta(t_i)\}$ by combining Definition 7 and (A.16), has a decreasing hazard rate. Since it is a monotone transformation of $\boldsymbol{Z}$, $\boldsymbol{Z}$ must have a decreasing hazard rate as well. It's inverse is increasing. In condition (E) the inverse hazard rate of $\boldsymbol{Z}$ is evaluated at $z = q + (N-1)y(b_i)$ which for any fixed b_i is increasing in q. Together this implies that $\left[\frac{1-F_Z(q+(N-1)y(b_i))}{f_Z(q+(N-1)y(b_i))}\right]$ is increasing in q for any fixed b_i. Since, in addition, $y'(b_i) < 0$ the first term of (E) increases in q. The second term of (E) is strictly decreasing in q given $v(\cdot, t_i)$ strictly decreases in q. Taken together this shows that the RHS is strictly increasing in q. $\square$

Part 2. *The optimum lies in the interior of the function space.*

To prove that the unique local maximum is a global maximum, I must verify that it cannot be optimal to choose a function that lies on the boundaries of the function space.[2] Corner solutions in variational calculus problems (or more generally optimal control theory) are points at which the functional (or Hamiltonian) is not differentiable. By assumption any function on which the functional depends is differentiable. There are no such points and with it no corner solutions. This completes the proof of Theorem 2. $\square$

A.3 Proof of Corollary 1

Let $v(q,t_i) = \max\{t_i - \rho q, 0\}$ with $\rho > 0$, assume $N > \frac{\overline{Q}\rho}{\underline{t}}$, and consider distributions of total supply on support $[0, \overline{Q} > 0]$ and types on $[\underline{t} > 0, \overline{t}]$ under which $\boldsymbol{q_i^*}$ is drawn from the Generalized Pareto Distribution $F_{q_i^*}(q) = 1 - \left[\frac{\sigma(\xi,t_i)+\xi q}{\sigma(\xi,t_i)}\right]^{-\frac{1}{\xi}}$ with scale parameter $\sigma(\xi,t_i) = -\xi\left(\frac{N(1-\xi)-1}{N(1-\xi)\rho}\right)(t_i - \underline{t}) - \xi\left(\frac{\overline{Q}}{N}\right)$ and shape parameter $\xi \in (-\infty, -1]$. The goal is to show that

$$b(q,t_i) = \begin{cases} \left(\frac{1}{1-\xi}\right)\left[t_i - \xi\underline{t}\right] - \left(\frac{\rho}{N(1-\xi)-1}\right)\left[(n-1)q - \xi\overline{Q}\right] & \text{for } q \in \left[0, -\left(\frac{\sigma(\xi,t_i)}{\xi}\right)\right] \\ v(q,t_i) & \text{for } q \in \left(-\left(\frac{\sigma(\xi,t_i)}{\xi}\right), \infty\right) \end{cases} \quad \text{(A.18)}$$

[2] This is analogous to finding global maxima when optimizing functions (not functionals) that are defined over a finite interval. From there we are used to checking that the value of the objective function at the boundaries of the interval is not higher than the value it takes at the interior maximum.

is a symmetric equilibrium. The proof is split into two parts, summarized in two lemmas.

The proof follows a guess and verify approach. I guess that there is such an equilibrium and compute how much an agent wins $\boldsymbol{q_i^*}$ when all play this strategy. With this I can use Theorem 2 to compute the bidding function that an agent submits when $\boldsymbol{q_i^*}$ follows the assumed Generalized Pareto Distribution. I verify that this function coincides with the guess (Lemma 7). If the hazard rate of $\boldsymbol{q_i^*}$ was decreasing, this would conclude the proof. Yet, the hazard rate of the Generalized Pareto Distribution with bounded support ($\xi < 0$) is increasing. The theorem does not apply directly. What remains to show is that the agent has indeed no profitable deviation because the necessary conditions which defined his bidding function are also sufficient (Lemma 8).

Lemma 7. *Function (A.18) fulfills the necessary conditions of a symmetric BNE.*

Proof. Assume all play equilibrium guess (A.18). By Theorem 2 a candidate function for a symmetric equilibrium takes the following form

$$b(q, t_i) = t_i - \rho q - \rho \int_q^{\overline{q}_i^*} \left[\frac{1 - F_{q_i^*}(x)}{1 - F_{q_i^*}(q)}\right]^{\frac{N-1}{N}} dx \text{ on } [0, \overline{q}_i^*].$$

To verify that the guess is a valid equilibrium candidate we must compute (i) the winning quantities and (ii) their support.

(i) To calculate how much an agent wins in equilibrium q_i^*, first, invert $b(\cdot, t_i)$ to determine the corresponding demand functions

$$x(p, t_i) = y(p) + \eta(t_i) \text{ with } y(p) \equiv a^* - e^* p \text{ and } \eta(t_i) \equiv c^* t_i \tag{1.5}$$

and equilibrium parameters

$$c^* = \frac{e^*}{(1-\xi)} = \left(\frac{N(1-\xi)-1}{\rho(N-1)(1-\xi)}\right) \tag{A.19}$$

$$e^* = \left(\frac{N(1-\xi)-1}{\rho(N-1)}\right) \tag{A.20}$$

$$a^* = \left(\frac{\xi}{N-1}\right)\left[\overline{Q} - \left(\frac{N(1-\xi)-1}{\rho(1-\xi)}\right)\underline{t}\right] \tag{A.21}$$

Now sum over all agents to obtain the market clearing price

$$\boldsymbol{Q} = x(\boldsymbol{p^c}, t_i) + \sum_{j \neq i} x(\boldsymbol{p^c}, \boldsymbol{t_j}) \overset{(1.5)}{=} Na^* + c^* t_i + c^* \sum_{j \neq i} \boldsymbol{t_j} - Ne^* \boldsymbol{p^c}$$

$$\Rightarrow \boldsymbol{p^c} = \frac{a^*}{e^*} + \frac{c^*}{Ne^*}\left[t_i + \sum_{j \neq i} \boldsymbol{t_j}\right] - \frac{\boldsymbol{Q}}{N}$$

and evaluate i's submitted demand at the clearing price:

$$\boldsymbol{q^*}(t_i) \equiv x(\boldsymbol{p^c}, t_i) = \frac{1}{N}\left[\boldsymbol{Q} - c^* \sum_{j \neq i} \boldsymbol{t_j} + (N-1)c^* t_i\right].$$

In the main text I call this amount $\boldsymbol{q_i^*}$ for notational ease.

(ii) The support $[\underline{q}^*(t_i), \overline{q}^*(t_i)]$ is given by the support of the total supply and the types. Since $c^* > 0$

$$\underline{q}^*(t_i) = \max\left\{0, \frac{1}{N}\left[0 - c^*(N-1)[\overline{t} - t_i]\right]\right\} \tag{A.22}$$

$$\overline{q}^*(t_i) = \frac{1}{N}\left[\overline{Q} - c^*(N-1)[\underline{t} - t_i]\right] \quad \text{with} \quad c^* = \left(\frac{N(1-\xi)-1}{\rho(N-1)(1-\xi)}\right) \tag{A.23}$$

Now, from basic statistics we know that the support of a standard Generalized Pareto Distribution with bounded support (i.e. with $\xi < 0$), location parameter $\alpha \in \mathbb{R}$, and scale parameter $\sigma > 0$ is $\left[\alpha, \alpha - \left(\frac{\sigma}{\xi}\right)\right]$. In my framework, where total supply may be 0, the lower bound of i's winning quantity, and with it the location parameter, is 0.[3] To see this, consider the lowest winning quantity the highest type $\overline{t}$ can achieve: $\underline{q}^*(\overline{t}) \overset{(A.22)}{=} \left[0 - c^*(N-1)[\overline{t} - \overline{t}]\right] = 0$. Since quantity is bounded below by 0, all lower types also win 0 in the worst case scenario: $\underline{q}^*(t_i) \overset{(A.22)}{=} 0$ for all t_i. The upper-bound of the support is, for all t_i, given by the scale parameter $\left(\frac{\sigma(\xi,t_i)}{\xi}\right)$. For any fixed ξ, it is easy to verify that the $\sigma(\xi,t_i) = -\xi\left(\frac{N(1-\xi)-1}{N(1-\xi)\rho}\right)(t_i - \underline{t}) - \xi\left(\frac{\overline{Q}}{N}\right)$ of the Corollary fulfills $\overline{q}^*(t_i) \overset{(A.23)}{=} -\left(\frac{\sigma(\xi,t_i)}{\xi}\right)$. The scale parameter $\sigma(\xi,t_i)$ is weakly positive since $\rho > 0, \xi < 0, N \geq 2, \overline{Q} > 0$.

(iii) Now I can use the bid-representation theorem to calculate the bidding function:

$$b(q,t_i) = t_i - \rho q - \rho \int_q^{\overline{q}_i^*} \left[\frac{1 - F_{q_i^*}(x)}{1 - F_{q_i^*}(q)}\right]^{\frac{N-1}{N}} dx$$

$$b(q,t_i) = \left(\frac{1}{1-\xi}\right)[t_i - \xi\underline{t}] - \left(\frac{\rho}{N(1-\xi)-1}\right)\left[(n-1)q - \xi\overline{Q}\right].$$

The solution coincides with the guess (A.18). Notice that it is strictly decreasing in quantity (since $\xi < 0, N \geq 2, \rho > 0$, $N(1-\xi) - 1 > 0$), and that condition $N > \frac{\overline{Q}\rho}{\underline{t}}$ ensures that

$$v(t_i, \overline{q}_i^*(t_i)) = \underline{t} + \left(\frac{t_i - \underline{t}}{(\xi-1)N}\right) - \frac{\overline{Q}}{N}\rho \geq 0 \;\forall t_i.$$

Therefore the bidding function never drops below 0. □

Lemma 8. *Function (A.18) is a BNE.*

Proof. The proof is analogous to the proof of Lemma 6 in the proof of the theorem. Here I only replace the parts which rely on the assumption that the distribution of i's winning quantity has a decreasing hazard rate (Auxiliary Lemma 2 and Part 1 of Lemma 3). The hazard rate is now weakly increasing.

Auxiliary Lemma 4 (Analogous to Auxiliary Lemma 2)**.** *Function (A.18) fulfills the sufficient conditions of a local maximum on* $q \in [0, \overline{q}_i^*] = \left[0, -\left(\frac{\sigma(\xi,t_i)}{\xi}\right)\right]$.

[3] Notice that this is the easiest case, at it eliminates type-dependence of the lower bound of $\boldsymbol{q}^*(t_i)$.

Proof. Recall the sufficient condition for a local maximum from page 68:

$$-[v(q,t_i)-b(q,t_i)]\left\{(N-1)\left[\frac{\partial f_{q_i^*}(q)}{\partial q}\right]-f_{q_i^*}(q)\left[\frac{b''(q,t_i)}{b'(q,t_i)}\right]\right\}+2b'(q,t_i)f_{q_i^*}(q)<0. \qquad \text{(A.14)}$$

It is always satisfied when the hazard rate is decreasing, and might hold when the hazard rate is increasing. Inserting all functional forms of the corollary, the sufficient condition (A.14) becomes

$$\left(\frac{(N-1)N\rho^2(1-\xi)}{(N(1-\xi)-1)(-\xi)}\right)\left(\frac{1}{f_2(t_i)}\right)^{-\frac{1}{\xi}} f_1(q,t_i)^{-\left[\frac{1+\xi}{\xi}\right]}(-2+N(1+\xi))<0 \qquad \text{(A.24)}$$

$\forall q\in[0,\overline{q}^*(t_i)), t_i$ with

$$f_1(q,t_i)\equiv-(t_i-\underline{t}-(1-\xi)(\rho\overline{Q}-N(\rho q-t_i+\underline{t}))) \quad \text{decreasing in } q \text{ and increasing in } t_i$$
$$f_2(t_i)\equiv(t_i-\underline{t})(N(1-\xi)-1)+\overline{Q}\rho(1-\xi) \quad \text{increases in } t_i.$$

In what follows I show that this condition is fulfilled for $\xi\le-1$. To see this I simplify it. The first term is strictly positive since $\xi<0, N\ge2, \rho>0, \overline{Q}>0$.

$$\left(\frac{(N-1)N\rho^2(1-\xi)}{(N(1-\xi)-1)(-\xi)}\right)>0$$

Also the second term is strictly positive for all types, $\left(\frac{1}{f_2(t_i)}\right)^{-\frac{1}{\xi}}>0$. This is because

$$f_2(\underline{t})>0\Rightarrow f_2(t_i)>0\ \forall t_i \ \text{ since } f_2(\cdot) \text{ is increasing in } t_i.$$

Canceling the two terms that are strictly positive, the sufficient condition (A.24) becomes

$$f_1(q,t_i)^{-\left[\frac{1+\xi}{\xi}\right]}(-2+N(1+\xi))<0\ \forall q\in[0,\overline{q}^*(t_i)),t_i.$$

I must show that this inequality holds for all types and all relevant quantities. This is not straightforward because the LHS may increase or decrease in quantity and type, depending on the sign of the exponent $-\left[\frac{1+\xi}{\xi}\right]$ and $(-2+N(1+\xi))$. The remainder of this section shows that it holds for $\xi\in(-\infty,-1]$.[4]

If $\xi=-1$, the sufficient condition is always satisfied. If $\xi<-1$ the exponent is negative, i.e. $-\left[\frac{1+\xi}{\xi}\right]<0$. $\xi<-1$ further implies that $(-2+N(1+\xi))<0$, so that the sufficient condition simplifies to

$$f_1(q,t_i)^{-\left[\frac{1+\xi}{\xi}\right]}>0\ \forall q\in[0,\overline{q}^*(t_i)),t_i.$$

Since $f_1(q,t_i)^{-\left[\frac{1+\xi}{\xi}\right]}$ is increasing in q and decreasing in t_i, it is fulfilled for all $q\in[0,\overline{q}^*(t_i)),t_i$ when:

$$f_1(0,\overline{t})>0\Leftrightarrow\underline{t}>\overline{t}+(\overline{Q}\rho+N(\overline{t}-\underline{t})(\xi-1)\Leftrightarrow\overline{Q}\rho>-\left(\frac{(\overline{t}-\underline{t})(N(1-\xi)-1)}{1-\xi}\right).$$

This is always satisfied given $\overline{Q},\rho>0, N\ge2, \xi\le-1$. □

[4]For $\xi\in(-1,0)$ I could neither show that the sufficient condition holds nor that it cannot hold. Therefore my proof therefore does not contradict Ausubel et al. (2014) who allow for $\xi<\frac{N-1}{N}$.

Auxiliary Lemma 5 (Analogous to Part 1 of Lemma 3)**.** *There is a unique function that satisfies the necessary condition for a symmetric equilibrium.*

Proof. The proof is analogous to the proof of Lemma 3. I show that there is exactly one q for any fixed relevant price $b_i \equiv b_i(q, t_i)$ that guarantees that the necessary condition is fulfilled. With decreasing hazard rate this was the case because

$$0 = \left[\frac{1 - F_Z\left(q + (N-1)y(b_i)\right)}{f_Z(q + (N-1)y(b_i))}\right]\left(\frac{1}{(N-1)(-1)y'(b_i)}\right) - [v(q, t_i) - b_i] \qquad (E)$$

was strictly increasing in quantity for any fixed price b_i. Now it is strictly increasing. To show this, I will, as above go via the hazard rate of i's winning quantity, which is drawn from a GPD by assumption. It assumes the following hazard rate

$$\left[\frac{1 - F_{q^*}(q)}{f_{q^*(q)}}\right] = \sigma(\xi, \varepsilon) + q\xi.$$

By definition $\boldsymbol{Z} = N\boldsymbol{q_i^*} - (N-1)\eta(t_i)$ so that $F_Z(z) = F_{q_i^*}(q)$ and $f_Z(z) = N f_{q_i^*}(q)$ for any z in the support of $\boldsymbol{Z}$. Therefore

$$\left[\frac{1 - F_Z\left(z\right)}{f_Z(z)}\right] = \frac{1}{N}[\sigma(\xi, \varepsilon) + z\xi] \text{ for any } z \text{ in the support of } \boldsymbol{Z}. \qquad (A.25)$$

In particular at realization $z = q + (N-1)y(b_i)$

$$\left[\frac{1 - F_Z\left(q + (N-1)y(b_i)\right)}{f_Z(q + (N-1)y(b_i))}\right] = \frac{1}{N}[\sigma(\xi, \varepsilon) + [q + (N-1)y(b_i)]\xi]. \qquad (A.26)$$

By assumption all other player's than i play the equilibrium guess. We thus know that agents $j \neq i$ choose $y(b_i) = a^* - e^* b_i$ with equilibrium coefficients as defined above. In addition the true marginal willingness to pay is linear $v(q, t_i) = t_i - \rho q$. Taken all together, condition (E) becomes

$$0 = \frac{1}{N}[\sigma(\xi, \varepsilon) + [q + (N-1)[a^* - e^* b_i]]\xi]\left(\frac{1}{(N-1)(-1)(-e^*)}\right) - [t_i - \rho q - b_i].$$

Taking the derivative w.r.t. q, and simplifying one can show that the RHS is strictly increasing in q given $N \geq 2, \xi \leq -1, \rho > 0$ and $\sigma(\xi, t_i)$ as defined in the corollary. Therefore, there is at most one quantity q for this price b_i that makes the condition bind.

This completes the proof of Corollary 1. □

A.3.1 Corollary 1 vs. Proposition 7 in Ausubel et al. (2014)

The Corollary relates to Ausubel et al. (2014)'s Proposition 7. They derive the unique linear equilibrium in a pay-as-bid auction in an environment *without* private types, where bid offers may drop below 0 in equilibrium. In their set-up, all bidders draw the same type, here called $\underline{\boldsymbol{t}}$, and are uncertain about the total supply $\boldsymbol{Q}$. Both are drawn from a joint distribution $F(Q, \underline{t})$, which is commonly known and has non-degenerate support. Different to my framework with independent private types, the single type of all agents and total supply may be correlated. With slight adaptation to my framework, their proposition reads as follows:

Proposition 7 (Ausubel et al. (2014)) *Let per-capita supply for any value be distributed according to the Generalized Pareto distribution with* $\alpha = 0$, *i.e.* $F_{q^*}(q|\underline{t}) = 1 - \left(\frac{\sigma+\xi q}{\sigma}\right)^{-\frac{1}{\xi}}$. *In the unique linear equilibrium, the strategy of bidder i is*

$$b(q, \underline{t}) = \underline{t} - \left(\frac{\rho}{N(1-\xi)-1}\right)[(N-1)q + N\sigma] \quad \textit{for } \xi < \frac{N-1}{N}. \tag{A.27}$$

Notice that I impose a stricter bound on $\xi \leq -1$. This comes from a difference in how we verify sufficient conditions. Ausubel et al. (2014) rely on the Maximum Theorem for compact intervals according to which a maximum (and a minimum) exists. I instead derive sufficient conditions that guarantee that the critical function is a maximum. To compare their result to mine not that the support of $\boldsymbol{q^*} \equiv \frac{\boldsymbol{Q}}{N}$ is bounded when $\xi < 0$: $q \in \left[0, \frac{\overline{Q}}{N}\right] = \left[0, -\frac{\sigma}{\xi}\right]$. Using $\frac{\overline{Q}}{N} = -\frac{\sigma}{\xi}$ their bidding function becomes

$$b(q, \underline{t}) = \underline{t} - \left(\frac{\rho}{N(1-\xi)-1}\right)\left[(N-1)q - \xi\overline{Q}\right] \quad \text{for } \xi \leq -1. \tag{A.27}$$

It is identical to the function of my Corollary given all draw the lowest type $t_i = \underline{t}$.

A.4 Extension: Reserve Price

The main result extends to auctions with reserve prices with distributions that may have unbounded support. The following extension is stated for distributions with unbounded support. The case of bounded support is analogous.

Theorem 2b. *Consider distributions of total supply and types such that the amount an agent wins in the symmetric equilibrium* $\boldsymbol{q_i^*}$ *is drawn from distribution* $F_{q_i^*}$ *with weakly decreasing hazard rate on* $[0, \infty)$ *and strictly positive density on* $[0, \overline{q}^R]$, *with* $v(\overline{q}^R, t_i) = R$.

There exists a pure-strategy Bayesian Nash equilibrium in which bidders submit

$$b^*(q, t_i) = v(q, t_i) - \int_q^{\overline{q}^R} \left[\frac{1 - F_{q_i^*}(x)}{1 - F_{q_i^*}(q)}\right]^{\frac{N-1}{N}} (-1)\left(\frac{\partial v(q, t_i)}{\partial q}\right) dx \qquad \textit{for } q \in [0, \overline{q}^R] \tag{1.1”}$$

provided the resulting equilibrium demand schedule is additively separable in the type on $[0, R]$, *and* $b^*(q, t_i) = v(q, t_i)$ *for* $q \in \left(\overline{q}^R, \infty\right)$.

The analogy with the first-price auction carries over. To see this, note that an agent of type s, drawn iid from $F_s(s)$ in a first-price auction with reserve price $R > 0$ chooses

$$\beta^*(s) = v(s) - \int_{s^R}^{s} \left[\frac{F(x)}{F(s)}\right]^{N-1} \left(\frac{\partial v(x)}{\partial s}\right) dx \qquad \textit{for } s \in [s^R, \overline{S}] \tag{1.1b''}$$

where $v(s) = s$ and $v(s^R) = R$, and 0 otherwise. Comparing the bidding function in the first-price auction to the one in the pay-as-bid auction in the way I illustrate in the main text, one can derive the main result for this extension. One only needs to replace $\overline{q}_i^*$ by $\overline{q}^R$/ 0 by s^R. Both bounds are now determined by the reserve price instead of the upper/lower bound of the distribution.

Proof. The proof is analogous to the proof of Theorem 2. Here I only highlight what changes. The support in Definition 6 is now $[0, \overline{q}^R]$, no longer $[0, \overline{q}_i^c]$. The difference is that the upper bound of the support is now exogenously given by the reserve price R. The bidder chooses this function so as to maximize his expected total surplus.

$$\max_{b_i(\cdot,t_i)} \mathcal{V}(b_i(\cdot, t_i)) \text{ s.t. } b_i(0, t_i) = \overline{p} < \infty \text{ and } b_i(\overline{q}^R, t_i) = R \tag{M}$$

where

$$\mathcal{V}(b_i(\cdot, t_i)) = \int_0^{\overline{q}^R} \left[\int_0^q v(x, t_i) - b_i(x, t_i)dx\right] g_i(q, b_i(q, t_i))dq.$$

The derivation of the Euler-Lagrange equation goes through without problems. One obtains the following necessary condition

$$0 = -\left(\frac{\partial G_i(q, b_i)}{\partial b_i}\right) [v(q, t_i) - b_i] - [1 - G_i(q, b_i)] \text{ for } q \in [0, \overline{q}^R] \tag{E'}$$

together with the end-point condition: $b_i(\overline{q}^R, t_i) = R$. Analogous to above, the next step is to evaluate the Euler-Lagrange Equation at the symmetric solution. The difference now is that i's winning quantity $\boldsymbol{q_i^*}$ has unbounded support $[0, \infty)$. If there was no positive reserve price $R > 0$, one would need to deal with improper integrals. Showing that they converge is not trivial. With a positive reserve price, however, the distribution that actually matters for the agent is bounded. Following Pycia and Woodward (2017) I construct the following auxiliary distribution

$$F_{q_i}^R(q) = \begin{cases} F_{q_i^*}(q) & \text{for } q < \overline{q}^R \\ 1 & \text{for } q \geq \overline{q}^R. \end{cases}$$

With it the Euler-Lagrange Equation can be expressed exactly in the way it was expressed above (N'). The difference is that it now holds for $q \in [0, \overline{q}^R)$ instead of $q \in [0, \overline{q}_i^*)$.

$$[v(q, t_i) - b(q, t_i)] = -\left[\frac{1 - F_{q_i}^R(q)}{f_{q_i}^R(q)}\right] b'(q, t_i) \text{ for } q \in [0, \overline{q}^R) \tag{N^R}$$

Analogous to above one can show that a function which fulfills the necessary condition is a global maximum. The solution to this differential equation for $q \in [0, \overline{q}^R)$ is

$$b(q,t_i) = [F^R_{q_i}(q) - 1]^{-[\frac{N-1}{N}]} \left[C + \left[\frac{N-1}{N}\right] \int_{q_0}^{q} [F^R_{q_i}(x) - 1]^{-[\frac{1}{N}]} f^R_{q_i}(x) v(x,t_i) dx \right] \text{ with } q_0 \in [0, \overline{q}^R).$$

The trick now is to choose C so that the solution goes through the end-point, i.e. $b(\overline{q}^R, t_i) = R$. For this approach $\overline{q}^R$ from the left and solve

$$R = \lim_{y \to \overline{q}^R -} [F^R_{q_i}(y) - 1]^{-[\frac{N-1}{N}]} \left[C + \left[\frac{N-1}{N}\right] \lim_{y \to \overline{q}^R -} \int_{q_0}^{y} [F^R_{q_i}(x) - 1]^{-[\frac{1}{N}]} f^R_{q_i}(x) v(x,t_i) dx \right].$$

By definition $\lim_{y \to \overline{q}^R -} F^R_{q_i}(y) = F(\overline{q}^R)$ so that

$$C = [F_{q_i^*}(\overline{q}^R) - 1]^{-[\frac{N-1}{N}]} - \left[\frac{N-1}{N}\right] \int_{q_0}^{\overline{q}^R} [F_{q_i^*}(x) - 1]^{-[\frac{1}{N}]} f_{q_i^*}(x) v(x,t_i) dx.$$

Inserting C into the bidding function and simplifying gives

$$b(q,t_i) = [F_{q_i^*}(q) - 1]^{-[\frac{N-1}{N}]} \left[R[F_{q_i^*}(\overline{q}^R) - 1]^{[\frac{N-1}{N}]} - \left[\frac{N-1}{N}\right] \int_{q}^{\overline{q}^R} [F_{q_i^*}(x) - 1]^{-[\frac{1}{N}]} f_{q_i^*}(x) v(x,t_i) dx \right].$$

Integrate by parts to simplify the function

$$\frac{N-1}{-N} \int_{q}^{\overline{q}^R} [F_{q_i^*}(x) - 1]^{\frac{1}{-N}} f_{q_i^*}(x) v(x,t_i) dx = -[F_{q_i^*}(x) - 1]^{\frac{N-1}{N}} v(x,t_i) \Big|_q^{\overline{q}^R} + \int_{q}^{\overline{q}^R} [F_{q_i^*}(x) - 1]^{\frac{N-1}{N}} v'(x,t_i) dx$$

Different to above $F_{q_i^*}(\overline{q}^R) \neq 1$, but strictly smaller.

$$= -[F_{q_i^*}(\overline{q}^R) - 1]^{[\frac{N-1}{N}]} v(\overline{q}^R, t_i) + [F_{q_i^*}(q) - 1]^{[\frac{N-1}{N}]} v(q,t_i) + \int_{q}^{\overline{q}^R} [F_{q_i^*}(x) - 1]^{[\frac{N-1}{N}]} v'(x,t_i) dx$$

The bidding function becomes for for $q \in [0, \overline{q}^R)$

$$b(q,t_i) = \mathcal{M}(q,R) + v(q,t_i) - \int_{q}^{\overline{q}^R} \left[\frac{1 - F_{q_i^*}(x)}{1 - F_{q_i^*}(q)}\right]^{[\frac{N-1}{N}]} (-1) v'(x,t_i) dx$$

with mark-up

$$\mathcal{M}(q,R) \equiv [R - v(\overline{q}^R, t_i)] \left[\frac{1 - F_{q_i^*}(\overline{q}^R)}{1 - F_{q_i^*}(q)}\right]^{[\frac{N-1}{N}]} = 0 \text{ by definition of } \overline{q}^R = v^{-1}(R, t_i).$$

We are left with

$$b(q,t_i) = v(q,t_i) - \int_{q}^{\overline{q}^R} \left[\frac{1 - F_{q_i^*}(x)}{1 - F_{q_i^*}(q)}\right]^{[\frac{N-1}{N}]} (-1) v'(x,t_i) dx \text{ for } q \in [0, \overline{q}^R]$$

where I extended the domain to include the point $\overline{q}^R$.

To complete the proof one can show as above that the solution fulfills the properties that I have assumed to derive it. □

Appendix B

Chapter 2

B.1 Proof of Theorem 3 and Lemma 2

Take the perspective of bidder i and let all other agents $j \neq i$ play as in equilibrium. Their bidding functions are denoted $\{b^*_{j,1}(\cdot), b^*_{j,2}(\cdot)\}$ with corresponding demand schedules $\{x^*_{j,1}(\cdot), x^*_{j,2}(\cdot)\}$. The proof proceeds in two main steps. The first part characterized bidder i's best reply to all others. The second solves for the symmetric equilibrium. Throughout I drop subscript i for notational convenience whenever it is unambiguous, and restrict attention to differentiable, strictly decreasing bidding functions that map from $[0, \overline{Q}_m] \to \mathbb{R}_+$. Others may not be submitted by the rules of the auction.

As a starting point, I must set up bidder i's maximization problem. Optimizing his expected total surplus from winning $\{\boldsymbol{q^c_1}, \boldsymbol{q^c_2}\}$ when offering prices $\{p_1(\boldsymbol{q^c_1}), p_2(\boldsymbol{q^c_2})\}$

$$\mathcal{V}(p_1(\cdot), p_2(\cdot)) \equiv \mathbb{E}\left[V(\boldsymbol{q^c_1}, \boldsymbol{q^c_2}) - \sum_{m=1,2} \int_0^{\boldsymbol{q^c_m}} p_m(q_m) dq_m\right] \text{ with } \boldsymbol{q^c_m} = \boldsymbol{Q_m} - \sum_{j \neq i} x^*_{j,m}(p_m(\boldsymbol{q^c_m})) \quad (\mathcal{V})$$

bidder i solves a problem of variational calculus subject to two end-point conditions. One of them comes from the feature that there is a highest amount that i can possibly win at market clearing.

Definition 8. *The maximal amount bidder i can achieve at market clearing when submitting $p_m(\cdot)$ is given by $\overline{q}^c_m = \overline{Q}_m - \sum_{j \neq i} x^*_{j,m}(p_m(\overline{q}^c_m))$ for $m = 1, 2$. Quantities $q_m \in [0, \overline{q}^c_m)$ are referred to as 'relevant' quantities. The upper bound of 'relevant equilibrium' quantities are denoted by $\overline{q}^*_{i,m}$ in any equilibrium, and $\overline{q}^*_m$ in a symmetric equilibrium.*

At $\overline{q}^c_m$ bidder i has no incentive to submit anything other than the (true) marginal benefit he expects from winning $\overline{q}^c_m$. The upper end-point condition is $p_m(\overline{q}^c_m) = \mathbb{E}[v_m(\overline{q}^c_m, \boldsymbol{q^c_{-m}}) | \overline{q}^c_m]$ for $m = 1, 2, -m \neq m$. The lower end-point guarantees that the bidding function is bounded from above. It says that the bid price for $q_m = 0$ must some arbitrary finite price $\overline{p}_m < 0$.

Taken together the maximization problem reads as follows

$$\max_{p_1(\cdot),p_2(\cdot)} \mathcal{V}(p_1(\cdot), p_2(\cdot))$$
$$\text{s.t. } p_m(0) = \overline{p}_m \text{ and } p_m(\overline{q}^c_m) = \mathbb{E}[v_m(\overline{q}^c_m, \boldsymbol{q^c_{-m}})|\overline{q}^c_m] \text{ for } m = 1, 2, -m \neq m. \quad \text{(MP)}$$

The proof splits into a sequence of lemma. Lemma 2, 9 and 10. The first is stated in the main text on page 24. Proven in Sections B.1.1 - B.1.3 the lemma characterize necessary and sufficient conditions for a global maximum of bidder i's optimization problem, assuming that the solution is twice differentiable.

Lemma 2 (Necessary Conditions)**.** *A BNE in which agent i submits $\{p^*_1(\cdot), p^*_2(\cdot)\}$ and all others $\{p^*_{j,1}(\cdot), p^*_{j,2}(\cdot)\}^N_{i\neq j}$ must for all $\{q_1, q_2\}$ and $i \in N$ satisfy*

$$[\mathbb{E}[v_m(q_m, \boldsymbol{q^*_{i,-m}})|q_m] - p^c_m] \left(\frac{\partial Pr\left(RS_m(p^c_m) \geq q_m\right)}{\partial p_m}\right) = Pr\left(RS_m(p^c_m) \geq q_m\right) \tag{2.9}$$

for $m = 1, 2; -m \neq m$, and clear both markets $p^c_m = p^*_m(q_m)$.

Lemma 9 (Sufficient Conditions)**.** *Consider a pair $\{p^*_1(\cdot), p^*_2(\cdot)\}$ that is twice differentiable and let $m = 1, -m = 2$. If at any q_1, q_2 of relevant quantities where $p^c_1 = p^*_1(q_1)$ and $p^c_2 = p^*_2(q_2)$ the following condition*

$$[v_1(q_2, \overline{Q}_2) - p^c_1]\, \frac{\partial^2 Pr(RS_1(p^c_1) \geq q_1)}{\partial^2 p_1} - 2\frac{\partial Pr(RS_1(p^c_1) \geq q_1))}{\partial p_1} - \int_0^{\overline{Q_2}} v(q_1, q_2)\frac{\partial^2 Pr(RS_1(p^c_1) \geq q_1 \text{ and } RS_2(p^c_2) \geq q_2)}{\partial^2 p_1} dq_2 \leq 0 \tag{S_1}$$

and

$$v(q_1, q_2)\frac{\partial^2 Pr(RS_1(p^c_1) \geq q_1 \text{ and } RS_2(p^c_2) \geq q_2)}{\partial p_1 \partial p_2} \leq 0 \tag{S_2}$$

as well as its analogue for $m = 2, -m = 1$ is satisfied, $\{p^*_1(\cdot), p^*_2(\cdot)\}$ is a local maximum.

Lemma 10. *$\{p^*_1(\cdot), p^*_2(\cdot)\}$ that locally maximizes the objective functional is a global solution.*

Taken together Lemma 2, 9, 10 characterize the best reply of bidder i to all others choosing an equilibrium strategy. The second part of the proof derives the symmetric equilibrium. When imposing symmetry across bidders the necessary condition becomes a linear differential equation, and the local sufficient conditions of Lemma 9 prove to hold (Lemma 11 proven in Section B.1.4). The final step is to derive the equilibrium bidding functions by solving the linear differential equation, to show that there is a unique such function, and that this function fulfills the assumed properties (Lemma 12 proven in Section B.1.5).

Lemma 11 (NC and SC under Symmetry)**.** *(i) If all bidders choose the same functions $\{p^*_1(\cdot), p^*_2(\cdot)\}$, the necessary condition of Lemma 2 simplifies to*

$$\mathbb{E}\left[v_m\left(q_m, \boldsymbol{q^*_{-m}}\right)|q_m\right] - p^*_m(q_m) = -\left[\frac{N}{N-1}\right]\left[\frac{1 - F_{q^*_m}(q_m)}{f_{q^*_m}(q_m)}\right] p^{*'}_m(q_m) \tag{N'}$$

for $q_m \in [0, \overline{q}^*_m)$ and both goods $m = 1, 2, m \neq -m$.
(ii) The (local) sufficient conditions of Lemma 9 are satisfied at the symmetric solution.

Lemma 12 (Existence and Uniqueness)**.** (*i*) *There is a unique function that satisfies the necessary and sufficient conditions (N′)*

$$p^*_m(q_m) = \mathbb{E}\left[v_m\left(q_m, \boldsymbol{q^*_{-m}}\right)|q_m\right] + \int_{q_m}^{\overline{q}^*_m} \left(\frac{\partial \mathbb{E}\left[v_m\left(x, \boldsymbol{q^*_{-m}}\right)|x\right]}{\partial q_m}\right)\left[\frac{1 - F_{q^*_m}(x)}{1 - F_{q^*_m}(q_m)}\right]^{\frac{N-1}{N}} dx. \tag{2.10}$$

on the domain of relevant equilibrium quantities $[0, \overline{q}^*_m]$.
(ii) It is twice differentiable and strictly decreasing.

B.1.1 Proof of Lemma 2

Step 1. The first step in solving i's maximization problem is summarized in an auxiliary lemma. It re-states the objective functional: $\mathcal{V}(p_1(\cdot), p_2(\cdot)) \equiv \mathbb{E}\left[V(\boldsymbol{q^c_1}, \boldsymbol{q^c_2}) - \sum_{m=1,2}\int_0^{\boldsymbol{q^c_m}} p_m(q_m)dq_m\right]$, using the following notation.

Definition 9. *Let bidder i submit functions s.t. $p_1(q_1) = p_1$ and $p_2(q_2) = p_2$. Denote*

$$G(q_1, q_2, p_1, p_2) \equiv \Pr\left(RS_1(p_1) \leq q_1 \text{ and } RS_2(p_2) \leq q_2\right) \tag{B.1}$$

and

$$G_m(q_m, p_m) = \Pr\left(RS_m(p_m) \leq q_m\right) \text{ for } m = 1, 2 \tag{B.2}$$

with residual supply $RS_m(p_m) = Q_m - \sum_{j\neq} x^*_{j,m}(p_m)$ as in Definition 2.
Refer to the corresponding joint and marginal density functions by g and g_m.

Auxiliary Lemma 6.

$$\mathcal{V}(p_1(\cdot), p_2(\cdot)) = \int_0^{\overline{Q}_1}\int_0^{\overline{Q}_2} F(q_1, q_2, p_1(q_1), p_2(q_2))dq_1dq_2 \tag{$\mathcal{V}$}$$

with

$$\begin{aligned}\mathcal{F}(q_1, q_2, p_1(q_1), p_2(q_1)) = &\sum_{m=1,2}\frac{1}{\overline{Q}_{-m}}[v_m(q_m, \overline{Q}_{-m}) - p_m(q_m)][1 - G_m(q_m, p_m(q_m))]\\ &- v(q_1, q_2)[1 - G(q_1, q_2, p_1(q_1), p_2(q_1))]\end{aligned} \tag{$\mathcal{F}$}$$

Proof. With Definition 9 the objective functional reads

$$\mathcal{V}(p_1(\cdot), p_2(\cdot)) = \int_0^{\overline{q}^c_2}\int_0^{\overline{q}^c_1}\left[V(q_1, q_2) - \sum_m \int_0^{q_m} p_m(x)dx\right] g(q_1, q_2, p_1(q_1), p_2(q_2))dq_1dq_2.$$

Notice that the bounds of integration vary in i's bid choices, because the supports of i's clearing price quantities depend on the price he offers for these amounts $p_m(\boldsymbol{q^c_m})$ for $m = 1, 2$. This is inconvenient because we will be looking for the *optimal* bid choices. Luckily there is a way around this complication. Since $g_m(q_m, p_m(q_m)) = g(q_1, q_2, p_1(q_2), p_2(q_2)) = 0$ for *any* bid price offers at $q_m \notin [0, \overline{q}^c_m]$ in either $m = 1$ or 2, I can extend the bounds of the integrals to $\overline{Q}_1$ and $\overline{Q}_2$:

$$\mathcal{V}(p_1(\cdot), p_2(\cdot)) = \int_0^{\overline{Q}_2}\int_0^{\overline{Q}_1}\left[V(q_1, q_2) - \sum_m \int_0^{q_m} p_m(x)dx\right] g(q_1, q_2, p_1(q_1), b_2(q_2))dq_1dq_2.$$

The next step is to re-express the objective functional in terms of distribution functions, such as $G(\cdot,\cdot,p_1(\cdot),p_2(\cdot))$, rather than densities $g(\cdot,\cdot,p_1(\cdot),p_2(\cdot))$. Who is not interested in following all mathematical steps may skip the rest of the proof of the auxiliary lemma and jump to page 82 (Step 2).

(a) Re-expressing the expected utility. Start by integrating the inner integral by parts, taking the derivative of $V(q_1,q_2)$ and integrating $g(q_1,q_2,p_1(q_1),p_2(q_2))$ w.r.t. q_1. Evaluate the first term at its bounds of integration und use $\int_0^0 g(q_1,q_2,p_1(q_1),p_2(q_2)) = 0$ and $\int_0^{\overline{Q}_1} g(q_1,q_2,p_1(q_1),p_2(q_2)) = \int_0^{\overline{q}_1^c} g(q_1,q_2,p_1(q_1),p_2(q_2)) = g(q_2,p_2(q_2))$ to obtain

$$\int_0^{\overline{Q}_2}\int_0^{\overline{Q}_1} V(q_1,q_2)g(q_1,q_2,p_1(q_1),p_2(q_2))dq_1dq_2 = \underbrace{\int_0^{\overline{Q}_2} V(\overline{Q}_1,q_2)g_2(q_2,p_2(q_2))dq_2}_{\equiv A} \\ - \underbrace{\int_0^{\overline{Q}_2}\left[\int_0^{\overline{Q}_1}\left[v_1(q_1,q_2)\int_0^{q_1} g(q_1,q_2,p_1(q_1),p_2(q_2))dq_1\right]dq_1\right]dq_2}_{\equiv B}.$$

Consider term A: Integrate by parts w.r.t. q_2, evaluate the first term at the bounds of integration and use $G_2(\overline{Q}_2,p_2(\overline{Q}_2)) = 1$ and $G_2(0,p_2(0)) = 0$ for all $p_2(\cdot)$ to obtain

$$A = V(\overline{Q}_1,\overline{Q}_2) - \int_0^{\overline{Q}_2} v_2(\overline{Q}_2,q_2)G_2(q_2,p_2(q_2))dq_2. \tag{A}$$

Now consider term B: Apply Fubini's Theorem to revert the order of integration of the two outer integrals, and integrate the inner integral corresponding to dq_2 by parts. Once more evaluate the first term at the bounds of integration and use that for any $p_1(\cdot),p_2(\cdot)$, $G(q_2,\overline{Q}_2,p_1(q_1),p_2(\overline{Q}_2)) = G_1(q_1,p_1(q_1))$ and $G(q_1,0,p_1(q_1),p_2(0)) = 0$ to achieve

$$B = \int_0^{\overline{Q}_1} v_1(q_1,\overline{Q}_2)G_1(q_1,p_1(q_1))dq_1 - \int_0^{\overline{Q}_2} v(q_1,q_2)G(q_1,q_2,p_1(q_1),p_2(q_2))dq_2. \tag{B}$$

Combining A - B, and applying the Fundamental Theorem of Calculus one last time the expected utility reads with $m=1,2, m \neq -m$

$$\mathbb{E}\left[V(\boldsymbol{q_1^c},\boldsymbol{q_2^c})\right] = \sum_{m=1,2}\int_0^{\overline{Q}_m} v_m(q_m,\overline{Q}_{-m})[1-G_m(q_m,p_m(q_m))]dq_m \\ - \int_0^{\overline{Q}_1}\int_0^{\overline{Q}_2} v(q_1,q_2)[1-G(q_1,q_2,p_1(q_1),p_2(q_2))]dq_2dq_1 \tag{EU}$$

(b) Re-expressing the expected payments.

$$\int_0^{\overline{Q}_m}\left[\int_0^{q_m} p_m(x)dx\right] g_m(q_m,p_m(q_m))dq_m = \left[\int_0^{q_m} p_m(x)dx\right] G_m(q_m,p_m(q_m))\Bigg|_0^{\overline{Q}_m} \\ - \int_0^{\overline{Q}_m} p_m(q_m)G_m(q_m,p_m(q_m))dq_m$$

Since $G_m(\overline{Q}_m,p_m(\overline{Q}_m)) = 1, G_m(0,p_m(0)) = 0$ for all $p_m(\cdot)$ this simplifies to

$$\int_0^{\overline{Q}_m}\left[\int_0^{q_m} p_m(x)dx\right] g_m(q_m,p_m(q_m))dq_m = \int_0^{\overline{Q}_m} p_m(q_m)[1-G_m(q_m,p_m(q_m))]dq_m \tag{EB_m}$$

→ **The objective function**

Combining (EU) -$\sum_{m=1,2}$ (EB_m) the objective function can be written as

$$\mathcal{V}(p_1(\cdot),p_2(\cdot)) = \sum_{m=1,2}\int_0^{\overline{Q}_m} \left[v_m(q_m,\overline{Q}_{-m}) - p_m(q_m)\right]\left[1-G_m(q_m,p_m(q_m))\right]dq_m$$
$$-\int_0^{\overline{Q}_1}\int_0^{\overline{Q}_2} v(q_1,q_2)[1-G(q_1,q_2,p_1(q_1),p_2(q_2))]dq_2dq_1 \qquad (\mathcal{V})$$

Pulling everything insight the double integral gives the expression of the auxiliary lemma.

Step 2. The next step is to derive necessary conditions for the maximization problem:

$$\max_{p_1(\cdot),p_2(\cdot)} \mathcal{V}(p_1(\cdot),p_2(\cdot))$$
$$\text{s.t. } p_m(0)=\overline{p}_m \text{ and } p_m(\overline{q}^c_m) = \mathbb{E}[v_m(\overline{q}^c_m, \boldsymbol{q^c_{-m}})|\overline{q}^c_m] \text{ for } m=1,2, -m\neq m. \qquad \text{(MP)}$$

Before doing so, notice that $\mathcal{V}(p_1(\cdot),p_2(\cdot))$ assumes value 0 as soon as one quantity, say q_1 lies outside the set of relevant quantities because $G(q_1,q_2,p_1(q_1),p_2(q_2)) = G_1(q_1,p_1(q_1)) = 1$ for $q_1 \notin [0,\overline{q}^c_1)$. This implies that the bidder may choose any prices on amounts that cannot win at market clearing provided that the bidding function is differentiable and decreasing (i.e. in the set of functions that he is allowed to submit). For now I will ignore the end-point conditions. In the end I show that they are satisfied at the solution (the symmetric equilibrium).

Suppose that $\{p^*_1(\cdot),p^*_2(\cdot)\}$ is a solution to $\max_{p_1(\cdot),p_2(\cdot)} V(p_1(\cdot),p_2(\cdot))$. Fix two arbitrary differentiable and decreasing functions $\{\xi_1(\cdot),\xi_2(\cdot)\}$ that map from $[0,\overline{Q}_m]\to\mathbb{R}_+$. Construct the following 'varied' function $\binom{p^*_1(q_1)}{p^*_2(q_2)} + \varepsilon\binom{\xi_1(q_1)}{\xi_2(q_2)}$ around the extremal. It is is, for every ε, differentiable on $[0,\overline{Q}_1]\times[0,\overline{Q}_2]$. Since $\{p^*_1(\cdot),p^*_2(\cdot)\}$ is a maximum,

$$I(\varepsilon) \equiv \int_0^{\overline{Q}_1}\int_0^{\overline{Q}_2} \mathcal{F}\left(q_1,q_2,p^*_1(q_1)+\varepsilon\xi_1(q_1),p^*_2(q_2)+\varepsilon\xi_2(q_2)\right)dq_1dq_2 \qquad \text{(B.3)}$$

must take its maximum for $\varepsilon=0$. In the remainder of the proof I compute the first variation $\frac{d}{d\varepsilon}I(\varepsilon)$ and derive conditions that guarantee $\frac{d}{d\varepsilon}I(0)=0$.

$$\frac{d}{d\varepsilon}I(\varepsilon) = \int_0^{\overline{Q}_1}\int_0^{\overline{Q}_2}\xi_1(q_1)\left(\frac{\partial\mathcal{F}(q_1,q_2,p^*_1(q_1)+\varepsilon\xi_1(q_1),p^*_2(q_2)+\varepsilon\xi_2(q_2))}{\partial p_1(q_1)}\right)dq_1dq_2$$
$$+\int_0^{\overline{Q}_1}\int_0^{\overline{Q}_2}\xi_2(q_2)\left(\frac{\partial\mathcal{F}(q_1,q_2,p^*_1(q_1)+\varepsilon\xi_1(q_1),p^*_2(q_2)+\varepsilon\xi_2(q_2))}{\partial p_2(q_2)}\right)dq_2dq_1.$$

Evaluating at $\varepsilon=0$, applying Fubuni's Theorem to reverse the order of integration of the second integral, pulling $\xi_1(q_1),\xi_2(q_2)$ out of the inner integral, and abbreviating $p^*_m(q_m) = p^*_m, \xi_m(q_m)=\xi_m$ for $m=1,2$ the necessary condition reads

$$\frac{d}{d\varepsilon}I(0) = \int_0^{\overline{Q}_1}\xi_1\left[\int_0^{\overline{Q}_2}\left(\frac{\partial\mathcal{F}(q_1,q_2,p^*_1,p^*_2)}{\partial p_1}\right)dq_1\right]dq_2 + \int_0^{\overline{Q}_2}\xi_2\left[\int_0^{\overline{Q}_1}\left(\frac{\partial\mathcal{F}(q_1,q_2,p^*_1,p^*_2)}{\partial p_2}\right)dq_2\right]dq_1 = 0.$$

For $\frac{d}{d\varepsilon}I(0)=0$ for any $\xi_1(\cdot),\xi_2(\cdot)$ it must be that

$$\int_0^{\overline{Q}_2}\left(\frac{\partial\mathcal{F}(q_1,q_2,p^*_1,p^*_2)}{\partial p_1}\right)dq_2 = 0 \qquad \text{(B.4)}$$

and

$$\int_0^{\overline{Q}_1} \left(\frac{\partial \mathcal{F}(q_1, q_2, p_1^*, p_2^*)}{\partial p_2} \right) dq_1 = 0. \tag{B.5}$$

Consider the condition for $m = 1$: (B.4). The other condition (B.5) is analogous. Given $\mathcal{F}$'s functional form (B.4) rearranges to

$$-\left[v_1(q_1, \overline{Q}_2) - p_1^*\right] \left(\frac{\partial G_1(q_1, p_1^*)}{\partial p_1} \right) + \int_0^{\overline{Q}_2} v(q_1, q_2) \left(\frac{\partial G(q_1, q_2, p_1^*, p_2^*)}{\partial p_1} \right) dq_2 = \left[1 - G_1(q_1, p_1^*)\right].$$

To obtain the condition of the lemma, apply the Fundamental Theorem of Calculus to replace $v_1(q_1, \overline{Q}_2) = \int_0^{\overline{Q}_2} v(q_1, q_2) dq_2 + v_1(q_2, 0)$. In addition use $G_{2|1}(q_2, p_2^* | q_1, p_1^*) \left(\frac{\partial G_1(q_1, p_1^*)}{\partial p_1} \right) = \left(\frac{\partial G(q_1, q_2, p_1^*, p_2^*)}{\partial p_1} \right)$ to obtain

$$\left[v_1(q_1, 0) + \int_0^{\overline{Q}_2} v(q_1, q_2) \left[1 - G_{2|1}(q_2, p_2^* | q_1, p_1^*)\right] - p_1^* \right] (-1) \left(\frac{\partial G_1(q_1, p_1^*)}{\partial p_1} \right) = 1 - G_1(q_1, p_1^*).$$

In a final step integrate by parts, using $G_{2|1}(\overline{Q}_2, p_2(\overline{Q}_2) | q_1, p_1^*(q_1)) = 1$ and $G_{2|1}(0, p_2^*(0) | q_1, p_1^*(q_1)) = 0$ to verify that the first two terms are the conditional expectation of the partial utility $\mathbb{E}\left[v_1(q_1, \boldsymbol{q_{i,2}^*}) \middle| q_1 \right]$. The condition becomes

$$\left[\mathbb{E}\left[v_1(q_1, \boldsymbol{q_{i,2}^*}) \middle| q_1 \right] - p_1^* \right] (-1) \left(\frac{\partial G_1(q_1, p_1^*)}{\partial p_1} \right) = \left[1 - G_1(q_1, p_1^*)\right]. \tag{2.9}$$

This completes the proof. To match the notation of lemma and main text it suffices to replace $[1 - G_1(q_1, p_1^*)]$ by $Pr_1(RS_1(p_1^c) \geq q_1)$ and $(-1)\left(\frac{\partial G_1(q_1, p_1^*)}{\partial p_1} \right)$ by $\left(\frac{\partial Pr_1(RS_1(p_1^c) \geq q_1))}{\partial p_1} \right)$.
□

B.1.2 Proof of Lemma 9

Lemma 9 (Sufficient Conditions). *Consider a pair $\{p_1^*(\cdot), p_2^*(\cdot)\}$ that is twice differentiable and let $m = 1, -m = 2$. If at any q_1, q_2 of relevant quantities where $p_1^c = p_1^*(q_1)$ and $p_2^c = p_2^*(q_2)$ the following condition*

$$\left[v_1(q_2, \overline{Q}_2) - p_1^c\right] \frac{\partial^2 Pr(RS_1(p_1^c) \geq q_1)}{\partial^2 p_1} - 2 \frac{\partial Pr(RS_1(p_1^c) \geq q_1))}{\partial p_1} - \int_0^{\overline{Q_2}} v(q_1, q_2) \frac{\partial^2 Pr(RS_1(p_1^c) \geq q_1 \text{ and } RS_2(p_2^c) \geq q_2)}{\partial^2 p_1} dq_2 \leq 0 \tag{S_1}$$

and

$$v(q_1, q_2) \frac{\partial^2 Pr(RS_1(p_1^c) \geq q_1 \text{ and } RS_2(p_2^c) \geq q_2)}{\partial p_1 \partial p_2} \leq 0 \tag{S_2}$$

as well as its analogue for $m = 2, -m = 1$ is satisfied, $\{p_1^*(\cdot), p_2^*(\cdot)\}$ is a local maximum.

Proof. A pair $\{p_1^*(\cdot), p_2^*(\cdot)\}$ that satisfies the necessary conditions is a local maximum if the second variation $\frac{d}{d\varepsilon}\left(\frac{d}{d\varepsilon} I(\varepsilon) \right)$ evaluated at $\varepsilon = 0$ is negative for all $\xi_1(\cdot), \xi_2(\cdot)$. Taking the second derivative of $I(\varepsilon)$, dropping dependencies of q_1, q_2 for notational ease, and evaluating at $\varepsilon = 0$ gives

$$\frac{d}{d\varepsilon}\left(\frac{d}{d\varepsilon}I(0)\right)=$$
$$\int_0^{\overline{Q}_1}\int_0^{\overline{Q}_2}\left[\xi_1^2\frac{\partial^2\mathcal{F}(q_1,q_2,p_1^*,p_2^*)}{\partial^2 p_1}+\xi_1\xi_2\frac{\partial^2\mathcal{F}(q_1,q_2,p_1^*,p_2^*)}{\partial p_1\partial p_2}+\xi_2\xi_1\frac{\partial^2\mathcal{F}(q_1,q_2,p_1^*,p_2^*)}{\partial p_2\partial p_1}+\xi_2^2\frac{\partial^2\mathcal{F}(q_1,q_2,p_1^*,^p*_2)}{\partial^2 p_2}\right]dq_1dq_2\leq 0.$$

Splitting up the integral, a stricter necessary condition that guarantees $\frac{d}{d\varepsilon}\left(\frac{d}{d\varepsilon}I(0)\right)\leq 0$ for all $\xi_1(\cdot),\xi_2(\cdot)$ is

$$\xi_1^2\int_0^{\overline{Q_2}}\frac{\partial^2\mathcal{F}(q_1,q_2,p_1^*,p_2^*)}{\partial^2 p_1}dq_2\leq 0 \text{ and } \xi_2^2\int_0^{\overline{Q_1}}\frac{\partial^2\mathcal{F}(q_1,q_2,p_1^*,p_2^*)}{\partial^2 p_2}dq_1\leq 0 \text{ and } \xi_1\xi_2\frac{\partial^2\mathcal{F}(q_1,q_2,p_1^*,p_2^*)}{\partial p_1\partial p_2}\leq 0$$

point-wise for all $\xi_1(q_1),\xi_2(q_1)$ at all q_1,q_2. Given that $\xi_1(\cdot),\xi_2(\cdot)$ map into $\mathbb{R}_+$ by assumption

$$\Leftrightarrow\int_0^{\overline{Q_2}}\frac{\partial^2\mathcal{F}(q_1,q_2,p_1^*,p_2^*)}{\partial^2 p_1}dq_2\leq 0 \text{ and } \int_0^{\overline{Q_1}}\frac{\partial^2\mathcal{F}(q_1,q_2,p_1^*,b_2^*)}{\partial^2 p_2}dq_1\leq 0 \text{ and } \frac{\partial^2\mathcal{F}(q_1,q_2,p_1^*,p_2^*)}{\partial p_1\partial p_2}\leq 0. \quad \text{(B.6)}$$

Given $\mathcal{F}$'s functional form

$$\int_0^{\overline{Q_2}}\frac{\partial^2\mathcal{F}(q_1,q_2,p_1^*,p_2^*)}{\partial^2 p_1}dq_2=2\left(\frac{\partial G_1(q_1,p_1^*)}{\partial p_1}\right)-[v_1(q_2,\overline{Q}_2)-p_1^*]\left(\frac{\partial^2 G_1(q_1,p_1^*)}{\partial^2 p_1}\right)+\int_0^{\overline{Q_2}}v(q_1,q_2)\left(\frac{\partial^2 G(q_1,q_2,p_1^*,p_2^*}{\partial^2 p_1}\right)dq_2 \quad (S_1)$$

with an analogous expression for $m=2$, and

$$\frac{\partial^2\mathcal{F}(q_1,q_2,p_1^*,p_2^*)}{\partial p_1\partial p_2}=v(q_1,q_2)\left(\frac{\partial^2 G(q_1,q_2,p_1^*,p_2^*)}{\partial p_2\partial p_1}\right) \quad (S_2)$$

This completes the proof. To match the notation of lemma and main text it suffices to replace $[1-G_1(q_1,p_1^*)]$ by $Pr_1(RS_1(p_1^c)\geq q_1)$ and $(-1)\left(\frac{\partial G_1(q_1,p_1^*)}{\partial p_1}\right)$ by $\left(\frac{\partial Pr_1(RS_1(p_1^c)\geq q_1))}{\partial p_1}\right)$ and so on. □

B.1.3 Proof of Lemma 10

Lemma 10. $\{p_1^*(\cdot),p_2^*(\cdot)\}$ *that locally maximizes the objective functional is a global solution.*

Proof. The proof consists of two parts. First I show that there is at most one set of functions $\{p_1^*(\cdot),p_2^*(\cdot)\}$ within the class of strictly decreasing, differentiable functions that satisfies the necessary conditions for a local maximum. Then I show that a pair $\{p_1^*(\cdot),p_2^*(\cdot)\}$ that locally maximizes the objective functional is a global solution.

Step 1. Consider auction 1 and assume bidder i chooses as in the symmetric equilibrium $p_2^*(\cdot)$ in auction 2. From Lemma 2 we know that an optimal bid in auction 1 must for all relevant quantities $q_1\in[0,\overline{q}_1^c)$ satisfy

$$[\mathbb{E}[v_1(q_1,\boldsymbol{q_2^*})|q_1]-p_1^*]-\left(\frac{Pr\left(RS_1(p_1^*(q_1))\geq q_1\right)}{\frac{\partial Pr\left(RS_1(p_1^*)\geq q_1\right)}{\partial p_1}}\right)=0. \quad (2.9)$$

To show that there is a unique function that solves this necessary condition I must show that there is a unique bid price at each quantity point. Since all candidate functions are strictly decreasing this is equivalent to showing that there is a unique amount i demands for each possible bid price. By Definition 2

$$Pr_1\left(RS_1(p_1^*) \geq q_1\right) = Pr\left(\boldsymbol{Q_1} \leq q_1 + (N-1)x_1^*(p_1^*)\right). \tag{B.7}$$

The necessary condition is therefore equivalent to

$$[\mathbb{E}[v_1(q_1, \boldsymbol{q_2^*})|q_1] - p_1^*] - \left(\frac{1 - F_{Q_1}(q_1 + (N-1)x_1(p_1^*))}{f_{Q_1}(q_1 + (N-1)x_1(p_1^*))}\right)\left(\frac{1}{(N-1)(-1)x_1'(p_1^*)}\right) = 0. \tag{B.8}$$

There is one quantity point q_1 (if any) that makes the condition bind for any fixed p_1^* if the RHS is strictly decreasing in quantity q_1. By assumption $\mathbb{E}[v_1(q_1, \boldsymbol{q_2^*})|q_1]$ is decreasing in q_1. Furthermore $(-1)x_1'(p_1^*) > 0$. Consequently, the RHS is strictly decreasing in q_1 if

$$\frac{\partial}{\partial q_1}\left(\frac{1 - F_{Q_1}(q_1 + (N-1)x_1(p_1^*))}{f_{Q_1}(q_1 + (N-1)x_1(p_1^*))}\right) \geq 0.$$

This follows from the assumption that total supply is drawn from a (marginal) distribution with weakly decreasing hazard rate.

The analogous argument hold for auction 2. Taken together the argument implies there can at most be a set of functions that solves the necessary conditions.

Step 2. To show that a pair of functions that locally maximizes the objective functional $V(p_1(\cdot), p_2(\cdot))$ is a global maximum it suffices to show that the solution cannot lie at the boundaries of the functional set. This is the case as long as $V(p_1(\cdot), p_2(\cdot))$ is, for any pair $\{p_1(\cdot), p_2(\cdot)\}$, bounded from above. To show this, let q_1^o, q_2^o be the quantities at which the utility function achieves its maximum on $[0, \overline{Q}_1] \times [0, \overline{Q}_2]$. Since bids must be positive,

$$V(p_1(\cdot), p_2(\cdot)) \equiv \mathbb{E}\left[V(\boldsymbol{q_1^c}, \boldsymbol{q_2^c}) - \sum_{m=1,2} \int_0^{\boldsymbol{q_m^c}} p_m(x)dx\right] \leq V(q_1^o, q_2^o) \text{ for any } \{p_1(\cdot), p_2(\cdot)\}. \quad \square$$

B.1.4 Proof of Lemma 11

Lemma 11 (NC and SC under Symmetry)**.** *(i) If all bidders choose the same functions $\{p_1^*(\cdot), p_2^*(\cdot)\}$, the necessary condition of Lemma 2 simplifies to*

$$\mathbb{E}\left[v_m\left(q_m, \boldsymbol{q_{-m}^*}\right)|q_m\right] - p_m^*(q_m) = -\left[\frac{N}{N-1}\right]\left[\frac{1 - F_{q_m^*}(q_m)}{f_{q_m^*}(q_m)}\right] p_m^{*'}(q_m) \tag{N'}$$

for $q_m \in [0, \overline{q}_m^*)$ and both goods $m = 1, 2, m \neq -m$.
(ii) The (local) sufficient conditions of Lemma 9 are satisfied at the symmetric solution.

Proof. When all bidders choose the same strictly decreasing and differentiable functions, they share the total supply quantities equally at market clearing, $\boldsymbol{q_m^*} = \frac{\boldsymbol{Q}_m}{N}$ in $m = 1, 2$. The necessary and sufficient conditions can be simplified by expressing (2.9) and its analogue for $m = 2$ in terms of the underlying joint and marginal distribution of total supply F, F_{Q_m}, instead of G_m and G: By market clearing $Q_m = Nq_m = \left(q_m + \sum_{j\neq i} x_m^*(p_m^*(q_m))\right)$ for $m = 1, 2$, it must be that

$$G(q_1, q_2, p_1^*(q_1), p_2^*(q_2)) = F\left(Nq_1, Nq_2\right) \tag{B.9}$$

$$G_m(q_m, p_m^*(q_m)) = F_{Q_m}\left(Nq_m\right) \quad \text{for any fixed } q_1 \in [0, \overline{q}_1^*],\ q_2 \in [0, \overline{q}_2^*]. \tag{B.10}$$

The first and second partial derivative of G, G_1 w.r.t. to the bid prices can be expressed as

$$\frac{\partial G(q_1, q_2, p_1^*(q_1), p_2^*(q_1))}{\partial p_1} = \left[\frac{\partial F\left(Nq_1, Nq_2\right)}{\partial Q_1}\right]\left(\sum_{j\neq i}\frac{\partial x_1^*(p_1^*(q_1))}{\partial p_1}\right) \tag{B.11}$$

$$\frac{\partial^2 G(q_1, q_2, p_1^*(q_1), p_2^*(q_2))}{\partial^2 p_1} = \left[\frac{\partial^2 F\left(Nq_1, Nq_2\right)}{\partial^2 Q_1}\right]\left(\sum_{j\neq i}\frac{\partial x_1^*(p_1^*(q_1))}{\partial p_1}\right)^2 + \left[\frac{\partial F\left(Nq_1, Nq_2\right)}{\partial Q_1}\right]\left(\sum_{j\neq i}\frac{\partial^2 x_1^*(p_1^*(q_1))}{\partial^2 p_1}\right) \tag{B.12}$$

$$\frac{\partial^2 G(q_1, q_2, p_1^*(q_1), p_2^*(q_2))}{\partial p_1 \partial p_2} = \left[\frac{\partial F\left(Nq_1, Nq_2\right)}{\partial Q_1 \partial Q_2}\right]\left(\sum_{j\neq i}\frac{\partial x_1^*(p_1^*(q_1))}{\partial p_1}\right)\left(\sum_{j\neq i}\frac{\partial x_2^*(p_2^*(q_2))}{\partial p_2}\right) \tag{B.13}$$

$$\frac{\partial G_1(q_1, p_1^*(q_1))}{\partial p_1} = f_{Q_1}\left(Nq_1\right)\left(\sum_{j\neq i}\frac{\partial x_1^*(p_1^*(q_1))}{\partial p_1}\right) \tag{B.14}$$

$$\frac{\partial^2 G_1(q_1, p_1^*(q_1))}{\partial^2 p_1} = \left(\frac{\partial f_{Q_1}\left(Nq_1\right)}{\partial Q_1}\right)\left(\sum_{j\neq i}\frac{\partial x_1^*(p_1^*(q_1))}{\partial p_1}\right)^2 + f_{Q_1}\left(Nq_1\right)\left(\sum_{j\neq i}\frac{\partial^2 x_1^*(p_1^*(q_1))}{\partial^2 p_1}\right) \tag{B.15}$$

Now, since all players i choose the same function $x_1^*(p_1) = (p_1^*)^{-1}(q_1)$, I can replace

$$\frac{\partial x_1^*(p_1^*(q_1))}{\partial p_1} = \frac{\partial (p_1^*)^{-1}(p_1^*(q_1))}{\partial p_1} = \left[\frac{1}{p_1^{*'}(q_1)}\right] \tag{B.16}$$

$$\frac{\partial^2 x_1^*(p_1^*(q_1))}{\partial^2 p_1} = \frac{\partial^2 (p_1^*)^{-1}(p_1^*(q_1))}{\partial^2 p_1} = -\left[\frac{p_1^{*''}(q_1)}{[p^{*'}(q_1)]^3}\right] \text{for all } j \neq i \tag{B.17}$$

by the Chain Rule and obtain

$$\frac{\partial G(q_1, q_2, p_1^*(q_1), p_2^*(q_2))}{\partial p_1} = (N-1)\left[\frac{\partial F\left(Nq_1, Nq_{-2})\right)}{\partial Q_1}\right]\left[\frac{1}{p_1^{*'}(q_1)}\right] \tag{B.18}$$

$$\frac{\partial^2 G(q_1, q_2, p_1^*(q_1), p_2^*(q_2))}{\partial^2 p_1} = (N-1)\left[(N-1)\left[\frac{\partial^2 F\left(Nq_1, Nq_2\right))}{\partial^2 Q_1}\right] - \left[\frac{\partial F\left(Nq_1, Nq_2\right)}{\partial Q_m}\right]\left[\frac{p_1^{*''}(q_1)}{p^{*'}(q_1)}\right]\right]\left[\frac{1}{p_1^{*'}(q_1)}\right]^2 \tag{B.19}$$

$$\frac{\partial^2 G(q_1, q_2, p_1^*(q_1), p_2^*(q_2))}{\partial p_1 \partial p_2} = (N-1) f(Nq_1, Nq_2)\left[\frac{1}{p_1^{*'}(q_1)}\right]\left[\frac{1}{p_2^{*'}(q_2)}\right] \tag{B.20}$$

$$\frac{\partial G_1(q_1, p_1^*(q_1))}{\partial p_1} = (N-1) f_{Q_1}\left(Nq_1\right)\left[\frac{1}{p_1^{*'}(q_1)}\right] \tag{B.21}$$

$$\frac{\partial^2 G_1(q_1, p_1^*(q_1))}{\partial^2 p_1} = (N-1)\left[(N-1)\left(\frac{\partial f_{Q_1}\left(Nq_1\right)}{\partial Q_1}\right) - f_{Q_1}\left(Nq_1\right)\left[\frac{p_1^{*''}(q_1)}{p^{*'}(q_1)}\right]\right]\left[\frac{1}{p_1^{*'}(q_1)}\right]^2 \tag{B.22}$$

Conditions for good 2 are analogous.

Statement (i) With (B.10) and (B.21) the necessary condition (2.9) for good 1 becomes

$$[\mathbb{E}\left[v_1(q_1, \boldsymbol{q_2^*})|\, q_1\right] - p_1^*(q_1)](-1) f_{Q_1}(Nq_1)(N-1)\left[\frac{1}{p_1^{*'}(q_1)}\right] = [1 - F_{Q_1}(Nq_1)] \tag{B.23}$$

or equivalently given $F_{Q_1}(Nq_1) = F_{q_1^*}(q_1)$ and $f_{Q_1}(Nq_1) = \left[\frac{1}{N}\right] f_{q_1^*}(q_1)$

$$[\mathbb{E}\left[v_1(q_1, \boldsymbol{q_2^*})|\, q_1\right] - p_1^*(q_1)](-1)\left[\frac{1}{N}\right] f_{q_1^*}(q_1)(N-1)\left[\frac{1}{p_1^{*'}(q_1)}\right] = [1 - F_{q_1^*}(q_1)]. \tag{B.24}$$

It rearranges to the expression of the Lemma

$$\mathbb{E}\left[v_1(q_1, \boldsymbol{q_2^*})|\, q_1\right] - p_1^*(q_1) = -\left[\frac{N}{N-1}\right]\left[\frac{1 - F_{q_1^*}(q_1)}{f_{q_1^*}(q_1)}\right] p_1^{*'}(q_1) \text{ for } q_1 \in [0, \overline{q}_1^*). \tag{N'}$$

The analogous holds for good 2.

Statement (ii). Consider the first sufficient condition (S_1). Lengthly tedious but straightforward derivations, where I substitute out for all G, G_1, G_2 and their partial- as well as cross partial derivatives, and insert the assumed functional form $v_1(q_1, q_2) = w_1(q_1) - \delta q_2$ rearrange (S_1) to

$$0 \geq N w_1'(q_1) - \delta\left[\frac{\partial \mathbb{E}[\boldsymbol{q_2^*}|q_1]}{\partial q_1}\right] \qquad \text{for } q_m = \overline{q}_m^* \tag{S_B'}$$

and

$$\frac{d}{dx}\ln\left[\frac{1 - F_{q_1^*}(x)}{f_{q_1^*}(x)}\right]\Bigg|_{x=q_1} \geq \left[\frac{N w_1'(q_1) - \delta\left[\frac{\partial \mathbb{E}[\boldsymbol{q_2^*}|q_1]}{\partial q_1}\right]}{\mathbb{E}[v_1(q_1, \boldsymbol{q_2^*}|q_1) - p_1^*(q_1)}\right] \qquad \text{for } q_1 \in [0, \overline{q}_1^*) \tag{S'}$$

The RHS is strictly negative negative for all $q_1 \in [0, \overline{q}_1^*]$ by the assumption that $\mathbb{E}[v_1(q_1, \boldsymbol{q_2^*}|q_1)]$ and $w_1(\cdot)$ are strictly decreasing. The sufficient condition thus holds at the end-point $\overline{q}_m^*$. For lower quantities, where $[\mathbb{E}[v_1(q_1, \boldsymbol{q_2^*}|q_1)] - p_1^*(q_1)] > 0$, it is fulfilled when the LHS of (S') is weakly positive. This is always the case by the assumption that total supply is drawn from a marginal distribution with weakly decreasing hazard rate.

Now consider the second (S_2). With (B.20) it becomes

$$\frac{\partial^2 F(q_1, q_2, p_1^*(q_1), p_2^*(q_2))}{\partial p_1 \partial p_2} = v(q_1, q_2)(N-1)^2 f(Nq_1, Nq_2)\left[\frac{1}{p_1^{*'}(q_1)}\right]\left[\frac{1}{p_2^{*'}(q_2)}\right] \leq 0.$$

Since $N \geq 2$, $p_1^*(\cdot)$, $p_2^*(\cdot)$ are strictly decreasing and $f(Nq_1, Nq_2) > 0$ by assumption, the condition simplifies to

$$\frac{\partial^2 F(q_1, q_2, p_1^*(q_1), p_2^*(q_2))}{\partial p_1 \partial p_2} = v(q_1, q_2) = -\delta \leq 0 \tag{B.25}$$

which holds by assumption. □

B.1.5 Proof of Lemma 12

Lemma 12 (Existence and Uniqueness)**.** *(i) The unique functions* $\{p_1^*(\cdot), p_2^*(\cdot)\}$ *that satisfies the necessary and sufficient conditions for a local maximum are*

$$p_m^*(q_m) = \mathbb{E}\left[v_m\left(q_m, \boldsymbol{q^*_{-m}}\right)|q_m\right] + \int_{q_m}^{\overline{q}_m^*} \left(\frac{\partial \mathbb{E}\left[v_m\left(x, \boldsymbol{q^*_{-m}}\right)|x\right]}{\partial q_m}\right) \left[\frac{1-F_{q_m^*}(x)}{1-F_{q_m^*}(q_m)}\right]^{\frac{N-1}{N}} dx. \quad (2.10)$$

on $q_m \in [0, \overline{q}_m^*]$ *for* $m = 1, 2, -m \neq m$.
(ii) Both are twice differentiable and strictly decreasing.

Proof. When condition (N') holds for all $q_m \in [0, \overline{q}_m^*] \equiv \left[0, \frac{\overline{Q}_m}{N}\right]$ it becomes a linear differential equation. Its unique solution is function (2.10) of the theorem. This solution is determined in three steps. In part 1, I first solve the differential equation for $q_m \in \left[0, \frac{\overline{Q}_m}{N}\right)$. I do so under the assumption that the solution $p(\cdot)$ is twice differentiable, bounded and strictly decreasing. Then I consider the boundary point $q_m = \frac{\overline{Q}_m}{N}$. In part 2, I verify that the derived solution function indeed has the assumed properties. Finally I check that no bid offer drops below zero.

For convenience I will drop subscript m and superscript *, and use

$$\tilde{v}(q) \equiv \mathbb{E}\left[v\left(q, \boldsymbol{q^*_{-m}}\right)|q\right].$$

Statement (i) Let $q \in \left[0, \frac{\overline{Q}}{N}\right)$ and assume $p(\cdot)$ is is twice differentiable, bounded and strictly decreasing on this domain. To simplify notation, let I refer to the hazard rate of the equilibrium quantity in each market as

$$h(q) \equiv \left[\frac{f_q(q)}{1-F_q(q)}\right] \quad (B.26)$$

and to the cumulative hazard rate of the equilibrium quantity as

$$\mathcal{H}(q) \equiv \int_0^q h(x)dx. \quad (B.27)$$

Inserting (B.26) in the first-order condition (N') and rearranging I obtain

$$p'(q) + p(q)W(q) = R(q) \quad \text{with} \quad (B.28)$$

$$W(q) = -\left[\frac{N-1}{N}\right] h(q) \text{ and} \quad (B.29)$$

$$R(q) = -\left[\frac{N-1}{N}\right] \tilde{v}(q)h(q) \quad (B.30)$$

This is a linear differential equation which can be solved with help of an integrating factor $P(q)$. Define it as

$$P(q) \equiv e^{\int_0^q W(x)dx} = e^{-\left[\frac{N-1}{N}\right]\int_0^q h(q)dx} \overset{(B.27)}{=} e^{-\left[\frac{N-1}{N}\right]\mathcal{H}(q)} \quad (B.31)$$

Now multiply the differential equation (B.28) by $P(q)$.

$$P(q)p'(q) + P(q)p(q)W(q) = P(q)R(q) \tag{B.32}$$
$$\Leftrightarrow [P(q)p(q)]' = P(q)R(q) \text{ since } P(q)W(q) = P'(q). \tag{B.33}$$

Integrating both sides, applying the Fundamental Theorem of Calculus gives

$$P(q)p(q) = \int_0^q P(x)R(x)dx + C \tag{B.34}$$

To determine the constant C let $q \to \frac{\overline{Q}}{N}$.

$$\lim_{q\to\frac{\overline{Q}}{N}} P(q) = \lim_{q\to\frac{\overline{Q}}{N}} e^{-\left[\frac{N-1}{N}\right]\mathcal{H}(q)} = \lim_{q\to\frac{\overline{Q}}{N}} \frac{1}{e^{\left[\frac{N-1}{N}\right]\mathcal{H}(q)}} \to 0 \text{ because } \lim_{q\to\frac{\overline{Q}}{N}} \mathcal{H}(q) \to \infty \text{ due to } \lim_{q\to\frac{\overline{Q}}{N}} F_q(q) \to 1$$

Since $p(\cdot)$ is bounded by assumption, this implies that the RHS goes to 0. Therefore

$$C = -\int_0^{\frac{\overline{Q}}{N}} P(x)R(x)dx \tag{B.35}$$

Summarizing the unique solution to differential equation (N') is

$$p(q) = \frac{\int_0^q P(x)R(x)dx + C}{P(q)} \tag{B.36}$$

Inserting for $P(z), R(z)$ for $z = x, q$ as defined in (B.31) and (B.30), and rearranging gives

$$\begin{aligned} p(q) = &\left[\frac{N-1}{N}\right] e^{\left[\frac{N-1}{N}\right]\mathcal{H}(q)} \int_0^{\frac{\overline{Q}}{N}} \left[\tilde{v}(x)\left[e^{-\left[\frac{N-1}{N}\right]\mathcal{H}(x)}\right] h(x)\right] dx \\ &- \left[\frac{N-1}{N}\right] e^{\left[\frac{N-1}{N}\right]\mathcal{H}(q)} \int_0^q \left[\tilde{v}(x)\left[e^{-\left[\frac{N-1}{N}\right]\mathcal{H}(x)}\right] h(x)\right] dx \end{aligned} \tag{B.37}$$

A tedious but straightforward simplification of this formula gives the function in the theorem. First I simplify both integrals. I do so by integration by parts, integrating $e^{-\left[\frac{N-1}{N}\right]\mathcal{H}(x)}h(x)$ and taking the derivative of $\tilde{v}(x)$. Here I am using again that $\lim_{q\to\frac{\overline{Q}}{N}} \mathcal{H}(q) \to \infty$. After further simplifications where I am substituting out for $\mathcal{H}(z)$ and $h(z)$, $z = x, q$ according to they definitions in (B.26) and (B.27) I obtain

$$p(q) = \tilde{v}(q) - \int_q^{\frac{\overline{Q}}{N}} \left[\frac{e^{-\left[\frac{N-1}{N}\right]\int_0^x \left[\frac{f_q(y)}{1-F_q(y)}\right]dy}}{e^{-\left[\frac{N-1}{N}\right]\int_0^q \left[\frac{f_q(y)}{1-F_q(y)}\right]dy}}\right](-1)\tilde{v}'(x)dx$$

The final step is to simplify the numerator and denominator of the integral. Taking the natural logarithm and applying the Fundamental Theorem of Calculus, we see that

$$e^{-\left[\frac{N-1}{N}\right]\int_0^z \frac{f_q(y)}{1-F_q(y)}dy} = [1 - F_q(z)]^{\frac{N-1}{N}} \text{for } z = x, q.$$

This gives the functional form of the theorem for $q \in \left[0, \frac{\overline{Q}}{N}\right)$

$$p(q) = \tilde{v}(q) - \int_q^{\frac{\overline{Q}}{N}} \left[\frac{1 - F_q(x)}{1 - F_q(q)}\right]^{\frac{N-1}{N}} (-1)\tilde{v}'(x)dx \tag{2.10}$$

Now consider the boundary point $q = \frac{\overline{Q}}{N}$. Since $F_q\left(\frac{\overline{Q}}{N}\right)$=1, the first-order condition directly gives us the solution for this point $p\left(\frac{\overline{Q}}{N}\right) = \tilde{v}\left(\frac{\overline{Q}}{N}\right)$. Because $\int_q^{\frac{\overline{Q}}{N}} ...dx = 0$ for $q = \frac{\overline{Q}}{N}$ we can extend the domain of function (2.10) to include the boundary point.

Statement (ii) To finalize the proof it remains to show that the solution $p(\cdot)$ indeed has the assumed properties.

Twice differentiability of $p(\cdot)$ follows immediately from the assumptions that $F(\cdot)$ and $\tilde{v}(\cdot)$ are twice differentiable. Moreover, since $\tilde{v}(\cdot)$ is bounded and strictly decreasing, $p(\cdot)$ is bounded by $p\,(0) = \tilde{v}\,(0)$. It remains to show that $p(q)$ is strictly decreasing in q. To do so, I take the derivative of bidding function (2.10), where I multiplied the minus sign out of the integral

$$p'(q) = \tilde{v}'(q) + \frac{d}{dq}\left\{\int_q^{\frac{\overline{Q}}{N}} \left[\frac{1 - F_q(x)}{1 - F_q(q)}\right]^{\frac{N-1}{N}} \tilde{v}'(x)dx\right\}$$

By the Leibniz-rule

$$p'(q) = \tilde{v}'(q) + \int_q^{\frac{\overline{Q}}{N}} \frac{\partial}{\partial q}\left\{\left[\frac{1 - F_q(x)}{1 - F_q(q)}\right]^{\frac{N-1}{N}} \tilde{v}'(x)\right\} dx - \left[\frac{1 - F_q(q)}{1 - F_q(q)}\right]^{\frac{N-1}{N}} \tilde{v}'(q)$$

The first and last term cancel out.

$$p'(q) = \int_q^{\frac{\overline{Q}}{N}} \frac{\partial}{\partial q}\left\{\left[\frac{1 - F_q(x)}{1 - F_q(q)}\right]^{\frac{N-1}{N}} \tilde{v}'(x)\right\} dx$$

Taking the partial derivative inside the integral I obtain

$$p'(q) = \int_q^{\frac{\overline{Q}}{N}} - \left[\frac{N-1}{N}\right] [1 - F_q(q)]^{-\left[\frac{N-1}{N}\right]-1} (-f_q(q))[1 - F_q(x)]^{\frac{N-1}{N}} \tilde{v}'(x)dx$$

which rearranges to

$$p'(q) = \left[\frac{N-1}{N}\right] f_q(q)\,[1 - F_q(q)]^{-\left[\frac{2N-1}{N}\right]} \int_q^{\frac{\overline{Q}}{N}} [1 - F_q(x)]^{\frac{N-1}{N}} \tilde{v}'(x)dx$$

Now, since $\left[\frac{N-1}{N}\right] f_q(q)\,[1 - F_q(q)]^{-\left[\frac{2N-1}{N}\right]} > 0$ for any $q \in [0, \frac{\overline{Q}}{N})$

$$p'(q) < 0 \;\text{ iff }\; \int_q^{\frac{\overline{Q}}{N}} [1 - F_q(x)]^{\frac{N-1}{N}} \tilde{v}'(x)dx < 0 \tag{B.38}$$

Because $1 - F_q(x) > 0$ and $\tilde{v}'(x) < 0$ on $x \in [0, \frac{\overline{Q}}{N})$ by assumption, this always holds. For $x = \frac{\overline{Q}}{N}$, $p(x) = \tilde{v}(x)$ which is strictly decreasing by assumption. In consequence, $p(q)$ is strictly decreasing on $\left[0, \frac{\overline{Q}}{I}\right]$.

There is one property that $p(\cdot)$ must fulfill in addition which I have not emphasized so far because I did not rely on it when solving the bidder's maximization problem: Bids must be non-negative. Since $p(\cdot)$ is strictly decreasing, it suffices to have

$$p\left(\frac{\overline{Q}}{N}\right) = \tilde{v}\left(\frac{\overline{Q}}{N}\right) \geq 0$$

This holds by Assumption 2 (ii). □

B.2 Proof of Corollary 2

Corollary 2 follows directly from Theorem 3 under the additional assumptions that the joint distribution is the FGM copula and marginal distributions are identical in both markets.

With $\tilde{v}_m(q_m) \equiv \mathbb{E}[v_m(q_m, \boldsymbol{q^*_{-m}})|q_m]) \overset{(2.2)}{=} w_m(q_m) - \delta\mathbb{E}[\boldsymbol{q^*_{-m}}|q_m]$, bidders choose

$$p_m(q_m) = \tilde{v}_m(q_m) + \int_{q_m}^{\frac{\overline{Q}_m}{N}} \left[\frac{1 - F_{q^*_m}(x)}{1 - F_{q^*_m}(q_m)}\right]^{\frac{N-1}{N}} \tilde{v}'_m(x)dx \text{ on } q \in \left[0, \overline{q}^*_m \equiv \frac{\overline{Q}_m}{N}\right]. \tag{2.10}$$

To compute this function, let us first determine $\tilde{v}_m(q_m)$'s functional form under the FGM. $\tilde{v}_m(q_m)$ depends on $\mathbb{E}\left[\boldsymbol{q^*_{-m}}\middle| q_m\right] = \left[\frac{1}{N}\right]\mathbb{E}\left[\boldsymbol{Q_{-m}}\middle| Q_m\right]$. Combining Schucany et al. (1978) (p. 651) and Crane and van der Hoek (2008) (p. 56)

$$\mathbb{E}\left[\boldsymbol{Q_{-m}}\middle| Q_m\right] = \mathbb{E}[\boldsymbol{Q_{-m}}] - \theta[1 - 2F_{Q_m}(Q_m)] \int F_{Q_{-m}}(Q_{-m})[1 - F_{Q_{-m}}(Q_{-m})dQ_{-m}$$

with $c_{-m} \equiv \int F_{Q_{-m}}(Q_{-m})[1 - F_{Q_{-m}}(Q_{-m})]dQ_{-m}$ and $\theta = \rho\left(\frac{\sigma_m\sigma_{-m}}{c_m c_{-m}}\right)$ so that

$$\mathbb{E}\left[\boldsymbol{Q_{-m}}\middle| Q_m\right] = \mathbb{E}[\boldsymbol{Q_{-m}}] - \rho\left(\frac{\sigma_m\sigma_{-m}}{c_m}\right)[1 - 2F_{Q_m}(Q_m)].$$

Therefore

$$\tilde{v}_m(q_m) = w_m(q_m) - \delta\left\{\mathbb{E}[\boldsymbol{q^*_{-m}}] - \rho\left(\frac{\sigma_m\sigma_{-m}}{c_m}\right)\left[\frac{1}{N}\right][1 - 2F_{q^*_m}(q_m)]\right\} \tag{B.39}$$

with derivative

$$\tilde{v}'_m(q_m) = w'_m(q_m) - \delta\rho\left(\frac{\sigma_m\sigma_{-m}}{c_m}\right)\left[\frac{1}{N}\right]2f_{q_m}(q_m). \tag{B.40}$$

With (B.39) and (B.40) equation (2.10) becomes

$$\begin{aligned} p_m(q_m) = w_m(q_m) - \delta\left\{\mathbb{E}[\boldsymbol{q^*_{-m}}] - \rho\left(\frac{\sigma_m\sigma_{-m}}{c_m}\right)\left[\frac{1}{N}\right][1 - 2F_{q^*_m}(q_m)]\right\} \\ - \int_{q_m}^{\frac{\overline{Q}_m}{N}} \left[\frac{1 - F_{q^*_m}(x)}{1 - F_{q^*_m}(q_m)}\right]^{\frac{N-1}{N}} \left[w'_m(x) + \delta\rho\left(\frac{\sigma_m\sigma_{-m}}{c_m}\right)\left[\frac{1}{N}\right]2f_{q_m}(x)\right]dx. \end{aligned}$$

Rearranging gives

$$p_m(q_m) = p_m^{SA}(q_m) - \delta D_m(q_m, \delta, \rho)$$

with

$$p_m^{SA}(q_m) = w_m(q_m) - \int_q^{\frac{\overline{Q}_m}{N}} \left[\frac{1 - F_{q_m^*}(x)}{1 - F_{q_m^*}(q_m)}\right]^{\frac{N-1}{N}} w_m'(x)dx$$

and

$$D_m(q_m, \delta, \rho) \equiv \delta \left\{\mathbb{E}[\boldsymbol{q^*_{-m}}] - \rho\left(\frac{\sigma_m \sigma_{-m}}{c_m}\right)\left[\frac{1}{N}\right]\mathcal{J}(q_m)\right\}$$

$$\text{where } \mathcal{J}(q_m) \equiv [1 - 2F_{q_m^*}(q_m)] - \int_{q_m}^{\frac{\overline{Q}_m}{N}} \left[\frac{1 - F_{q_m^*}(x)}{1 - F_{q_m^*}(q)}\right]^{\frac{N-1}{N}} 2f_{q_m}(x)dx.$$

According to Theorem 1, $p_m^{SA}(q_m)$ is the equilibrium function of a stand-alone auction with marginal willingness to pay of $w_m(q_m)$. Simplifying $\mathcal{J}(q_m)$ by integrating by parts and using $F_{q_m^*}\left(\frac{\overline{Q}_m}{N}\right) = 1$, the discount factor rearranges to the expression of the Corollary. To do so, compute the integral of $\mathcal{J}(q_m)$:

$$\begin{aligned}\int_{q_m}^{\frac{\overline{Q}_m}{N}} \left[\frac{1 - F_{q_m^*}(x)}{1 - F_{q_m^*}(q)}\right]^{\frac{N-1}{N}} 2f_{q_m}(x)dx &= 2[1 - F_{q_m^*}(q_m)]^{-\left[\frac{N-1}{N}\right]}\left[-\left[\frac{N}{2N-1}\right][1 - F_{q_m^*}(x)]^{\left[\frac{2N-1}{N}\right]}\Bigg|_{q_m}^{\frac{\overline{Q}_m}{N}}\right] \\ &= 2\left[\frac{N}{2N-1}\right][1 - F_{q_m^*}(q_m)]\end{aligned}$$

$\mathcal{J}(q_m)$ simplifies to

$$\begin{aligned}\mathcal{J}(q_m) &= [1 - 2F_{q_m^*}(q_m)] - 2\left[\frac{N}{2N-1}\right][1 - F_{q_m^*}(q_m)] \\ &= -\left[\frac{1}{2N-1}\right][1 + 2(N-1)F_{q_m^*}(q_m)]\end{aligned}$$

Inserting $\mathcal{J}(q_m)$ into $D_m(q_m, \delta, \rho)$ I obtain the expression as stated in Corollary 2:

$$D_m(q_m, \delta, \rho) = \delta\left\{\mathbb{E}[\boldsymbol{q^*_{-m}}] + \rho\left(\frac{\sigma_m \sigma_{-m}}{c_m}\right)\left[\frac{1}{N}\right]\left[\frac{1}{2N-1}\right]\left[1 + 2(N-1)F_{q_m^*}(q_m)\right]\right\}. \quad \square$$

B.3 Proof of Corollary 3

Fix some $q_m \in \left[0, \frac{\overline{Q}_m}{N}\right]$.

Statement (i). Let $\rho \neq 0$. The goal is to show that the term in curly brackets of $D_m(q_m, \delta, \rho)$ is positive:

$$\mathbb{E}[\boldsymbol{q^*_{-m}}] + \left(\frac{\sigma_m \sigma_{-m}}{c_m}\right)\left[\frac{1}{N}\right]\left[\frac{1}{2N-1}\right]\left[1 + 2(N-1)F_{q_m^*}(q_m)\right] > 0 \tag{B.41}$$

For $\rho > 0$, where all terms are positive, this is easy to see. To show the property for $\rho < 0$, define $\bar{\rho} \equiv -\rho$. I need to show that

$$N\mathbb{E}[\boldsymbol{q^*_{-m}}] > \bar{\rho}\left(\frac{\sigma_m \sigma_{-m}}{c_m}\right)\left[\frac{1}{2N-1}\right]\left[1 + 2(N-1)F_{q^*_m}(q_m)\right] \quad \text{(B.42)}$$

holds for all possible values of $\bar{\rho} \in \left(0, \left(\frac{c_m c_{-m}}{\sigma_m \sigma_{-m}}\right)\right]$ and all relevant quantities $q_m \in \left[0, \overline{Q}_m/N\right]$. This is the case if it holds for the values of $\bar{\rho}$ and q_m for which the RHS assumes its maximal value. These are $\bar{\rho} = \left(\frac{c_m c_{-m}}{\sigma_m \sigma_{-m}}\right)$ and $q_m = \overline{Q}_m/N$. Evaluating condition (B.42) at these values, using $F_{q^*_m}\left(\overline{Q}_m/N\right) = 1$, it remains to show that $N\mathbb{E}[\boldsymbol{q^*_{-m}}] > c_{-m} \Leftrightarrow \mathbb{E}[\boldsymbol{Q_{-m}}] > c_{-m}$. To see that this always holds, recall the definition of $c_{-m} \equiv \int_0^{\overline{Q}_m} F_{Q_{-m}}(y)[1 - F_{Q_{-m}}(y)]dy$. In addition express the expected value as $\mathbb{E}[\boldsymbol{Q_{-m}}] = \int_0^{\overline{Q}_m}[1 - F_{Q_{-m}}(y)]dy$:

$$\mathbb{E}[\boldsymbol{Q_{-m}}] > c_{-m} \Leftrightarrow \int_0^{\overline{Q}_m}[1 - F_{Q_{-m}}(y)]dy > \int_0^{\overline{Q}_m} F_{Q_{-m}}(y)[1 - F_{Q_{-m}}(y)]dy$$

$$\Leftrightarrow \int_0^{\overline{Q}_m}\left\{[1 - F_{Q_{-m}}(y)] - F_{Q_{-m}}(y)[1 - F_{Q_{-m}}(y)]\right\}dy > 0$$

$$\Leftrightarrow \int_0^{\overline{Q}_m}[1 - F_{Q_{-m}}(y)]^2 dy > 0 \text{ which holds trivially.}$$

Statement (ii) & (iii). Both properties follow immediately from the functional form of $D_m(q_m, \delta, \rho)$ in combination with property (i).

Statement (iv). The partial derivative of the discount factor w.r.t. ρ is

$$\frac{\partial D_m(q_m, \delta, \rho)}{\partial \rho} = \delta\left\{\left(\frac{\sigma_m \sigma_{-m}}{c_m}\right)\left[\frac{1}{N}\right]\left[\frac{1}{2N-1}\right]\left[1 + 2(N-1)F_{q^*_m}(q_m)\right]\right\}.$$

Since $c_m, \sigma_m, \sigma_{-m} > 0, N \geq 2$ it is positive only if $\delta > 0$.

Statement (v). Taking the derivative of $D_m(q_m, \delta, \rho)$ w.r.t. q_m gives

$$\frac{\partial D_m(q_m, \delta, \rho)}{\partial q_m} = \delta\rho\left\{\left(\frac{\sigma_m \sigma_{-m}}{c_m}\right)\left[\frac{1}{N}\right]\left[\frac{1}{2N-1}\right]2(N-1)f_{q_m}(q_m)\right\}.$$

By assumption the density of the total supply is strictly positive on its support. Therefore $f_{q_m}(q_m) > 0$ is strictly positive on $\left[0, \frac{\overline{Q}_m}{N}\right]$. Recalling that $c_m, \sigma_m, \sigma_{-m} > 0, N \geq 2$ property (v) follows. □

Appendix C

Chapter 3

C.1 Proofs

C.1.1 Proof of Proposition 1 and Corollary 5

Proposition 1. Recall that the dealer expects the following payoff from owning q_1, q_2

$$V(q_1, q_2, s^g_{i,\tau}) = U(q_1, q_2, s^g_{i,\tau}) + \mathbb{E}\left[revenue(\boldsymbol{x_1}, \boldsymbol{x_2}|q_1, q_2) - cost(\boldsymbol{x_1}, \boldsymbol{x_2}|q_1, q_2)\right] \tag{3.8}$$

with $revenue(x_1, x_2|q_1, q_2) = \sum_{m=1}^{2} p_m(x_1, x_2|q_1, q_2)x_m$. Given the aggregate inverse demand of the dealer's clients (3.5)

$$\begin{aligned} V(q_1, q_2, s^g_{i,\tau}) = U(s^g_{i,\tau}, q_1, q_2) \\ &+ \int_0^{\kappa_1 q_1}\int_0^{\kappa_2 q_2} [p_1(x_1, x_2)x_1 + p_2(x_2, x_1)x_2] f(x_1, x_2)dx_1 dx_2 \\ &+ \int_0^{\kappa_1 q_1}\int_{\kappa_2 q_2}^{1} [p_1(x_1)x_1 - \gamma x_2] f(x_1, x_2)dx_1 dx_2 + \int_{\kappa_1 q_1}^{1}\int_0^{\kappa_2 q_2} [p_2(x_2)x_2 - \gamma x_1] f(x_1, x_2)dx_1 dx_2 \\ &- \int_{\kappa_1 q_1}^{1}\int_{\kappa_2 q_2}^{1} [\gamma x_1 x_2] f(x_1, x_2)dx_1 dx_2. \end{aligned}$$

Inserting the assumed functional forms (3.4), (3.5), and $f(x_1, x_2) = 1 + 3\rho(1 - 2F_1(x_1)(1 - 2F_2(x_2))$ integrating and taking the partial derivative w.r.t. q_1 we obtain

$$\begin{aligned} v_1(q_1, q_2, s^g_{1,i,\tau}) =& 1/2\gamma\kappa_1(-1+\rho) - 2\gamma\kappa_1\kappa_2^3 q_2^3\rho + 1/2\gamma\kappa_1\kappa_2^2 q_2^2(1+3\rho) \\ &+ q_1^2(-6\gamma\kappa_1^3\kappa_2 q_2\rho + 3(3\gamma+2e)k_1^3 k_2^2 q_2^2\rho - 4(\gamma+2e)\kappa_1^3\kappa_2^3 q_2^3\rho + \kappa_1^3(-b+\gamma\rho)) \\ &+ q_1(2(3\gamma+2e)\kappa_1^2\kappa_2^3 q_2^3\rho + c\kappa_1^2\kappa_2 q_2(1+3\rho) + 1/2\kappa_1^2(2a+\gamma-3\gamma\rho) - 1/2\kappa_1^2\kappa_2^2 q_2^2(\gamma+2e+15\gamma\rho+6e\rho)) \\ &+ (1-\kappa_1)s^g_{1,i,\tau}. \end{aligned}$$

A Taylor expansion around $\left(\frac{1}{2}, \frac{1}{2}\right)$ gives

$$v_1(q_1, q_2, s^g_{1,i,\tau}) = (1-\kappa_1)s^g_{1,i,\tau} + h_0(\kappa_1, \kappa_2, \gamma, \rho) + h_1(\kappa_1, \kappa_2, \gamma, a, b, e, \rho)q_1 + h_2(\kappa_1, \kappa_2, e, \rho)q_2$$

with

$$
\begin{aligned}
h_0(\kappa_1,\kappa_2,\gamma,\rho) =& \frac{1}{16}(4b\kappa_1^3 + 2e\kappa_1^2\kappa_2^2(2+(6-9\kappa_1-6\kappa_2+8\kappa_1\kappa_2)\rho)) \\
&+\frac{1}{16}(\gamma\kappa_1(8(-1+\rho)+\kappa_1^2(-2+\kappa_2)(2+\kappa_2(-11+8\kappa_2))\rho) \\
&+\frac{1}{16}(\gamma\kappa_1(+2\kappa_2^2(-1-3\rho+4\kappa_2\rho)+2\kappa_1\kappa_2(-2+\kappa_2-3(-1+\kappa_2)(-2+3\kappa_2)\rho)) \\
h_1(\kappa_1,\kappa_2,\gamma,a,b,e,\rho) =& \frac{1}{8}\kappa_1^2(8a-8b\kappa_1-2e\kappa_2^2(1+(-1+2\kappa_1)(-3+2\kappa_2)\rho)) \\
&+\frac{1}{8}\kappa_1^2(\gamma(4+4\kappa_2-\kappa_2^2-(-2+\kappa_2)(-6+3\kappa_2-6\kappa_2^2+2\kappa_1(-2+\kappa_2)(-1+2\kappa_2))\rho)) \\
h_2(\kappa_1,\kappa_2,\gamma,e,\rho) =& -\frac{1}{4}\kappa_1 k_2(-2\gamma\kappa_1+\gamma(-2+\kappa_1)k_2+2e\kappa_1 k_2)(1+3(-1+\kappa_1)(-1+k_2)\rho) \quad \square
\end{aligned}
$$

Corollary 5. Securities become more complementary when $h_2(\kappa_1,\kappa_2,\gamma,e,\rho)$ increases. For any $\kappa_m \in [0,1]$ and any ρ that is within the allowed range of correlation parameters of the Farlie-Gumbel-Morgenstern Distributions with uniform marginal distributions, $[-1/3, 1/3]$:

$$
\begin{aligned}
\frac{\partial h_2(\kappa_1,\kappa_2,\gamma,e,\rho)}{\partial e} &= -(1/2)\underbrace{\kappa_1^2\kappa_2^2(1+3(-1+\kappa_1)(-1+\kappa_2)\rho)}_{\geq 0} \leq 0 \\
\frac{\partial h_2(\kappa_1,\kappa_2,\gamma,e,\rho)}{\partial \gamma} &= -(1/4)\underbrace{\kappa_1(\kappa_1(-2+\kappa_2)-2\kappa_2)}_{\leq 0}\underbrace{\kappa_2(1+3(-1+\kappa_1)(-1+\kappa_2)\rho)}_{\geq 0} \geq 0 \\
\frac{\partial h_2(\kappa_1,\kappa_2,\gamma,e,\rho)}{\partial \rho} &= -(1/4)\underbrace{\kappa_1(\kappa_1(-2+\kappa_2)-2\kappa_2)}_{\leq 0}\underbrace{\kappa_2(1+3(-1+\kappa_1)(-1+\kappa_2)\rho)}_{\geq 0} \geq 0 \quad \square
\end{aligned}
$$

C.1.2 Proof of Proposition 2

The proposition follows from Proposition 3 when all δ parameters are 0. □

C.1.3 Proof of Proposition 3

Take the perspective of bidder i who belongs to bidder group $g \in \{c, d\}$. Fix his type, a time slot τ, as well as one of his information sets $\theta^g_{i,\tau}$, and let all other agents $j \neq i$ play a type-symmetric equilibrium. In this equilibrium it must be optimal for the bidder to choose the same set of functions $\{b^g_1(\cdot, \theta^g_{i,\tau}), ...b^g_M(\cdot, \theta^g_{i,\tau})\}$ as all other bidders in his bidder group with information $\theta^g_{i,\tau}$. These M functions must *jointly* maximize the bidder's expected total surplus. It must therefore be the case that each of the functions $b^g_m(\cdot, \theta^g_{i,\tau})$ maximizes his expected total surplus *separately* when fixing all the other bidding functions $-m$ at the optimum. To determine necessary conditions of the type-symmetric equilibrium we can consequently fix the agent's strategy in all but one auction at the equilibrium. Without loss take this auction to be the one for security 1.

The remainder of the proof extends Kastl (2012)'s proof for a K-step equilibrium of a pay-as-bid auction that takes place in isolation without difficulties. To facilitate the comparison with the original proof (on pp. 347 - 348 of Kastl (2012)) we copy it as closely as possible but adopt the notation used in this article.

There are two main difference to the original proof. First, our framework allows bidders to update their bids due to arrival of new information. Such information arrives at discrete time slots $\tau = 1...\Gamma$. Bidding functions do not (only) depend on the bidder i's type $s^g_{i,\tau}$ drawn at time τ but on the (entire) information set he has at that time $\theta^g_{i,\tau}$. It includes the type, $s^g_{i,\tau} \subseteq \theta^g_{i,\tau}$. Since only final bids count, bidders bid as if it was their last bid each time they place a bid. We can just keep some τ fixed throughout the proof.

Second, following Hortaçsu and Kastl (2012) we allow for asymmetries in bidding behavior between dealers and customers. They draw types from (potentially) different distributions and may have different information available. The original proof extends to this set-up.

Simplified Notation. *We drop subscripts τ, i as well as superscript g.*
We refer to the amount a bidder with information θ wins at market clearing in auction m (for a given set of strategies in the event that τ is the time of the bidder's final bid) by $\boldsymbol{q_1^c}$, and the amount he wins in equilibrium by $\boldsymbol{q_1^*}$.

Notice that both, $\boldsymbol{q_1^c}$ and $\boldsymbol{q_1^*}$ are (for given strategies of all agents) functions of the total supply $\boldsymbol{Q_1}$ and the information of all agents $\{\boldsymbol{\theta_i}\}_{i=1}^N$. They are implicitly defined by market clearing.

The proof of the proposition relies on three lemmas. The second and third are taken from Kastl (2012).

Lemma 13. *Fix a bidder with information θ.*
Denote his marginal willingness to pay in auction m at step k when submitting some function $b_1'(\cdot, \theta)$ with $\{(b'_{1,k}, q'_{1,k-1}), (b'_{1,k+1}, q'_{1,k})\}$ by

$$\tilde{v}_1(q_1, \theta | b'_{1,k}, b'_{1,k+1}) \equiv \mathbb{E}\left[v_1\left(q_1, \boldsymbol{q_{-1}^*}, s_1\right) \middle| b'_{1,k} \geq \boldsymbol{P_1^c} > b'_{1,k+1}, \theta \right] \text{ for } q_1 \in (q'_{1,k-1}, q'_{1,k}]. \quad \text{(C.1)}$$

(i) $\tilde{v}_1(q_1, \theta | b'_{1,k}, b'_{1,k+1})$ is bounded.
(ii) In equilibrium, where the bidder submits function $b_1(\cdot, \theta)$ with $\{(b_{1,k}, q_{1,k-1}), (b_{1,k+1}, q_{1,k})\}$, $\tilde{v}_1(q_1, \theta | b_{1,k}, b_{1,k+1})$ is decreasing in q_1 and right-continuous in $b_{1,k}$.

Proof of Lemma 13. (*i*) By Assumption 3

$$\tilde{v}_1(q_1, \theta | b'_{1,k}, b'_{1,k+1}) \overset{(3.3)}{=} \alpha + (1 - \kappa_1)s_1 + \lambda_1 q_1 + \delta_1 \cdot \mathbb{E}\left[\boldsymbol{q_{-1}^*} | b'_{1,k} \geq \boldsymbol{P_1^c} > b'_{1,k+1}, \theta \right]$$

for $q_1 \in (q'_{1,k-1}, q'_{1,k}]$. Since types and total supply are drawn from distributions with bounded support by Assumptions 2 and 5, $\mathbb{E}\left[\boldsymbol{q_{-1}^*} | b'_{1,k} \geq \boldsymbol{P_1^c} > b'_{1,k+1}, \theta \right]$ and with it $\tilde{v}_1(q_1, \theta | b'_{1,k}, b'_{1,k+1})$ is bounded.

(ii) In equilibrium $\tilde{v}_1(q_1, \theta | b_{1,k}, b_{1,k+1})$ must be decreasing in q_1 or it could not give rise to a decreasing bidding function that fulfills the necessary conditions of Proposition 3.

To see why $\tilde{v}_1(q_1, \theta | b_{1,k}, b_{1,k+1})$ is right-continuous in $b_{1,k}$ note first that it can only jump discontinuously if changing $b_{1,k}$ breaks a tie between this bidder and at least one other bidder. Since there can be only countably many prices on which a tie might occur, however, there must exist a neighborhood at any $b_{1,k}$ for which for any price in that neighborhood there are no ties. Therefore, when perturbing b_k, there cannot be any discontinuous shift in the conditional probability measure and thus in the object of interest. □

Lemma 14. *Fix a bidder with information θ.*
If at some step k in auction 1, $\Pr(\boldsymbol{q_1^c} \geq q_{1,k} | \theta) > 0$*, then* $b_{1,k} \leq \tilde{v}_1(q_1, \theta | b_{1,k}, b_{1,k+1})$.

Proof of Lemma 14. The proof is analogous to Kastl (2012)'s proof of Lemma 2. It suffices to replace $v(q, s)$ in the original proof by $\tilde{v}_1(q_1, \theta | b_{1,k}, b_{1,k+1})$ and rely on Lemma 13. □

Lemma 15. (i) *Ties occur with zero probability for a.e. θ in any K-step equilibrium of simultaneous pay-as-bid auctions expect possibly at the last step ($k_1 = K_1$).*
(ii) *If a tie occurs with positive probability at the last step, a bidder with information θ must be indifferent between winning or losing all units between the lowest share he gets allocated after rationing in the event of a tie $\underline{q}_1^{RAT}$ and the last infinitesimal unit he may be allocated in equilibrium, $\overline{q}_1$:*

$$b_{1,K_1} = \tilde{v}_1(\bar{q}_1, \theta | b_{1,K_1}) \text{ where } \bar{q}_1 = \sup_{\{Q_1, \theta_{-i}\}} y_1(b_{1,K_1}, \theta | Q_1, \theta_{-i}) \; \forall q_1 \in [\underline{q}_1^{RAT}, \overline{q}_1].$$

Proof of Lemma 15. The proof is also analogous to the proof of Lemma 1 in Kastl (2012). In essence it suffices to replace the bidder's true valuation $v(q, s)$ in Kastl (2012) by $\tilde{v}_1(\cdot, \theta | b_k, b_{k+1})$ in equilibrium and $\tilde{v}_1(\cdot, \theta | b'_k, b'_{k+1})$ for deviations and rely on Lemma 13.

To facilitate this conversion, we demonstrate the beginning of the proof: Suppose that there exists an equilibrium, in which for a bidder i with information set θ a tie between at least two bidders can occur with positive probability $\pi_1 > 0$ in auction 1. Since there can be only finitely many prices that can clear the market with positive probability, in order for a tie to be a positive probability event, it has to be the case that there exists a positive measure subset of information sets $\hat{\Theta}_{-i} \in [0, 1]^{N-1}$ such that for some bidder j, and all profiles of information sets $\theta_{-i} \in \hat{\Theta}'_{-i} \subset \hat{\Theta}_{-i}$ (another positive measure subset) and some step k and l we have $b_{1,k}(\theta_i) = b_{1,l}(\theta_j) = P_1^c$. Without loss suppose that this event occurs at the bid $(b_{1,k}, q_{1,k})$, and that the maximum quantity allocated to i after rationing is $\bar{q}_1^{RAT} < q_{1,k}$. Let $\bar{S}_{1\pi}^R$ denote the maximal level of the residual supply at $b_{1,k}$ in the states leading to rationing at $b_{1,k}$.

Consider a deviation to a step $b'_{1,k} = b_{1,k} + \varepsilon$ and $q'_{1,k} = q_{1,k}$ where ε is sufficiently small. This deviation increases the probability of winning $q_{1,k} - q_{1,k-1}$ units. Most importantly in the states that led to rationing under the original bid, the bidder with information θ will now obtain $q_1^u > \bar{q}_1^{RAT}$ where $q_1^u \geq \min\{q_{1,k}, \bar{S}_{1\pi}^R\}$. Notice that since we hypothesized a positive probability of a tie at $b_{1,k}$, we need to have $q_{1,k-1} < \bar{q}_1^{RAT} < q_{1,k}$ due to rationing pro-rata on-the-margin. Therefore, the lower bound on the increase in θ's expected gross surplus from such a deviation is

$$ED_\varepsilon = \pi_1 \left(\tilde{V}_\varepsilon(q_1^u, \theta) - \tilde{V}(\bar{q}_1^{RAT}, \theta) \right) \tag{ED_ε}$$

where

$$\tilde{V}_\varepsilon(q_1^u, \theta) \equiv \int_0^{\bar{q}_1^{RAT}} \tilde{v}_1(q_1, \theta | b_1(q_1|\theta)) + \int_{\bar{q}_1^{RAT}}^{q_1^u} \tilde{v}_1(q_1, \theta | b'_{1,k}, b'_{1,k+1}) dq_1$$

and

$$\tilde{V}(\bar{q}_1^{RAT}, \theta) \equiv \int_0^{\bar{q}_1^{RAT}} \tilde{v}_1(q_1, \theta | b_1(q_1|\theta)) dq_1$$

with $\tilde{v}_1(q_1, \theta | b_1(q_1|\theta))$ denoting the true valuation when submitting $b_1(q_1|\theta)$ not just at step k, as $\tilde{v}_1(q_1, \theta | b_{1,k}, b_{1,k+1})$, but including all previous steps (if any).

To continue, let us first focus on steps other than the last one, $k < K_1$, and suppose that $\tilde{v}_1(\cdot, \theta | b_{1,k}, b_{1,k+1})$ is strictly decreasing. The increased bid $b_{1,k} + \varepsilon$ also results in an increase in the payment for the share requested at this step. This increase, however, is bounded by $(q_{1,k} - q_{1,k-1})\varepsilon$. Comparing the upper bound on the change in expected payment with the lower bound on the change in expected gross utility, in order for this deviation to be strictly profitable we need to obtain

$$(q_{1,k} - q_{1,k-1})\varepsilon < \pi_1 ED_\varepsilon. \tag{C.2}$$

As $b_{1,k} \leq \tilde{v}_1(q_{1,k}, \theta | b_{1,k}, b_{1,k+1})$ by Lemma 14 and $\tilde{v}_1(q_{1,k}, \theta | b_{1,k}, b_{1,k+1}) < \tilde{v}_1(q_1^u, \theta | b_{1,k}, b_{1,k+1})$, the LHS of (C.2) goes to 0 and the RHS to a strictly positive number as $\varepsilon \to 0$. Since $\tilde{v}_1(q_1, \theta | b_{1,k}, b_{1,k+1})$ is for any $q_1 \in [\bar{q}_1^{RAT}, q_{1,k}]$ right-continuous in $b_{1,k}$, the proposed deviation would indeed be strictly profitable for the bidder with information θ. Moreover, there can be only countable many θ's with a profitable deviation otherwise bidder i could implement this deviation jointly and thus for *a.e.* information sets θ ties have zero probability in equilibrium for all bidders i.

Relying on Lemma 13, the remainder of the proof is analogous to the original proof. It suffices to replace $v(q, s)$ by $\tilde{v}_1(\cdot, \theta | b_k, b_{k+1})$ in equilibrium and $\tilde{v}_1(\cdot, \theta | b'_k, b'_{k+1})$ when deviating, as well as $V(q^*, s) - V(\bar{q}_i^{RAT}, s)$ by ED_ε. In our environment with updating, a tie may occur with positive probability only at the last step and the bidder with information θ (at the previously fixed time τ) must not prefer winning any units in $\left[\underline{q}_1^{RAT}, \overline{q}_1\right]$ where $\overline{q}_1 = \sup_{\{Q_1, \theta_{-i}\}} y_1(b_{1,K_1}, \theta | Q_1, \theta_{-i})$ is the maximal quantity the bidder may be allocated in an equilibrium (in the event that τ is the time of his final bid). $\square$

Proof of Proposition 3. At step $k = K_1$ Lemma 14 specifies the optimal bid-choice. At steps $k < K_1$ Lemma 15 can be applied. Kastl (2012) pertubes the kth step to $q_1' = q_{1,k} - \epsilon$ and takes the limit as $q_1' \to q_{1,k}$. The original proof goes through without complications. It suffices to replace the type s by the information set θ, $\mathbb{E}\left[V(Q_i^c(Q, \boldsymbol{S}, \boldsymbol{y}(\cdot|S)), s_i)\right| \text{states}]$ by $\mathbb{E}\left[V(\boldsymbol{q_1^*}, \boldsymbol{q_{-1}^*}, s)\middle| \theta, \text{states}\right]$ with all states as specified in the original proof, and similarly $\mathbb{E}\left[V(Q_i^c(Q, \boldsymbol{S}, \boldsymbol{y}'(\cdot|S)), s_i)\right| \text{states}]$ by $\mathbb{E}\left[V(\boldsymbol{q_1^c}, \boldsymbol{q_{-1}^*}, s)\middle| \theta, \text{states}\right]$ where $\boldsymbol{q_1^c}$ denotes the amount the bidder wins at market clearing under the deviation in our simplified notation. □

C.2 Robustness

Robustness (1). Our main specification restricts the sample to bids submitted by dealers within 30 minutes prior to auction closure. Tables C.1 - C.3 display the estimation results of our main regression

$$\hat{v}_{t,m,i,\tau,k} = u_{t,m,i,\tau} + \lambda_m q_{t,m,i,\tau,k} + \delta_m \cdot \hat{q}^*_{t,-m,i,\tau,k} + \varepsilon_{t,m,i,\tau,k} \text{ for } m = 3M, 6M, 12M \qquad (3.12)$$

over the full time period (2002 - 2015) in various other specifications. In the first column we use all step-functions with more than one step, including customer and dealer bids. The second column restricts attention to all such bids submitted by dealers. In the the third to sixth column we narrow down the time period prior to auction closure, using bids by dealers only. All findings are robust to these changes. The same is true when splitting the sample into the time before the financial crisis, the crisis and the time afterwards.

Robustness (2). Tables C.4 - C.6 illustrate the robustness of our results with regards to the markup of 0.05 bsp (as lower bound), 0.5 bsp (as our main specification) and 5 bsp (as upper bound) and the restriction that $\tilde{v}_m(\cdot, s^g_{i,\tau}|\theta^g_{i,\tau})$ is decreasing (referred to as "Sorted" or "Unsorted" in the tables). Higher markups than 50 bsp are too large relative to the small amount by which a dealer shades one unit of a bill on average (< 0.25 bps in our main specification). We here show the estimation results when splitting the sample into the pre-crisis/crisis/and post-crisis periods as an example, using the sample of dealer bids submitted less than 30 minutes prior to auction closure. The same conclusions hold when using the full time period instead.

Table C.1: Robustness (1) 3M Auction

	all	dealers	≤1h	≤ 30min	≤ 10min	≤ 5min
λ_3M	-0.704***	-0.704***	-0.709***	-0.744***	-0.794***	-0.784***
	(-193.76)	(-189.47)	(-180.74)	(-165.83)	(-111.90)	(-82.97)
$\delta\_3M, 6M$	0.307***	0.307***	0.319***	0.334***	0.364***	0.299**
	(7.21)	(7.05)	(7.06)	(6.80)	(4.79)	(2.93)
$\delta\_3M, 1Y$	0.347***	0.348***	0.388***	0.355***	0.325***	0.359***
	(7.54)	(7.39)	(7.86)	(6.69)	(4.30)	(3.59)
Constant	995703.6***	995654.1***	995635.8***	995533.2***	995438.1***	995551.0***
	(3221068.68)	(3007762.84)	(2881372.45)	(2574206.36)	(1605259.59)	(1150196.70)
Observations	62100	58542	54856	45994	19176	9592

t statistics in parentheses, * $p < 0.05$, ** $p < 0.01$, *** $p < 0.001$

Table C.2: Robustness (1) 6M Auction

	all	dealers	≤1h	≤ 30min	≤ 10min	≤ 5min
λ_6M	-3.102***	-3.114***	-3.092***	-3.095***	-3.053***	-2.941***
	(-158.76)	(-155.55)	(-147.41)	(-132.97)	(-91.53)	(-66.09)
$\delta\_6M, 3M$	0.0551	0.0633	0.0533	0.00508	-0.0122	-0.0702
	(1.61)	(1.82)	(1.48)	(0.13)	(-0.21)	(-0.90)
$\delta\_6M, 1Y$	0.123	0.125	0.0430	0.0934	-0.185	-0.335
	(1.24)	(1.23)	(0.41)	(0.84)	(-1.19)	(-1.55)
Constant	991774.0***	991653.7***	991651.7***	991509.1***	991278.4***	991420.7***
	(1804001.79)	(1705631.07)	(1638079.00)	(1494437.31)	(1016043.88)	(727979.83)
Observations	44768	42282	39871	34267	15903	7907

t statistics in parentheses, * $p < 0.05$, ** $p < 0.01$, *** $p < 0.001$

Table C.3: Robustness (1) 12M Auction

	all	dealers	≤1h	≤ 30min	≤ 10min	≤ 5min
λ_1Y	-6.488***	-6.517***	-6.483***	-6.476***	-6.335***	-6.234***
	(-177.56)	(-175.89)	(-168.18)	(-154.51)	(-111.44)	(-86.29)
$\delta\_1Y, 3M$	-0.499***	-0.483***	-0.520***	-0.629***	-0.892***	-1.051***
	(-7.45)	(-7.15)	(-7.51)	(-8.51)	(-8.94)	(-8.20)
$\delta\_1Y, 6M$	0.816***	0.837***	0.806***	0.846***	0.862***	0.934**
	(4.53)	(4.60)	(4.32)	(4.34)	(3.42)	(2.88)
Constant	981740.1***	981611.8***	981560.9***	981316.6***	980497.0***	980254.1***
	(971863.17)	(936542.88)	(904564.02)	(841131.53)	(616248.14)	(475127.80)
Observations	52806	50410	48132	42830	23255	12921

t statistics in parentheses, * $p < 0.05$, ** $p < 0.01$, *** $p < 0.001$

Table C.4: Robustness (2) 3M Auction

Sorted	Pre (0.05)		Crisis (0.05)		Post (0.05)	
λ_3M	-0.669***	(-94.31)	-1.038***	(-97.69)	-0.585***	(-111.05)
$\delta\_3M, 6M$	0.229***	(3.95)	0.426***	(4.18)	0.384***	(5.31)
$\delta\_3M, 1Y$	0.316***	(5.38)	0.426***	(3.59)	0.335***	(4.43)
Constant	992189.1***	(1924060.25)	994345.9***	(1011800.72)	997786.5***	(2139360.74)
Sorted	Pre (0.5)		Crisis (0.5)		Post (0.5)	
λ_3M	-0.677***	(-95.27)	-1.044***	(-98.18)	-0.589***	(-111.95)
$\delta\_3M, 6M$	0.218***	(3.77)	0.420***	(4.12)	0.374***	(5.16)
$\delta\_3M, 1Y$	0.310***	(5.27)	0.419***	(3.54)	0.328***	(4.33)
Constant	992189.8***	(1920929.74)	994346.6***	(1011262.51)	997787.0***	(2139133.26)
Sorted	Pre (5)		Crisis (5)		Post (5)	
λ_3M	-0.727***	(-99.99)	-1.089***	(-101.64)	-0.620***	(-117.18)
$\delta\_3M, 6M$	0.140*	(2.37)	0.364***	(3.54)	0.287***	(3.94)
$\delta\_3M, 1Y$	0.260***	(4.33)	0.351**	(2.94)	0.260***	(3.42)
Constant	992194.4***	(1879696.08)	994352.2***	(1003982.65)	997790.7***	(2128431.94)
Unsorted	Pre (0.05)		Crisis (0.05)		Post (0.05)	
λ_3M	-0.648***	(-87.61)	-0.993***	(-89.62)	-0.565***	(-104.86)
$\delta\_3M, 6M$	0.181**	(3.00)	0.304**	(2.86)	0.362***	(4.89)
$\delta\_3M, 1Y$	0.240***	(3.93)	0.297*	(2.40)	0.327***	(4.22)
Constant	992188.3***	(1847434.30)	994343.5***	(970315.73)	997785.1***	(2091115.06)
Unsorted	Pre (0.5)		Crisis (0.5)		Post (0.5)	
λ_3M	-0.655***	(-88.26)	-0.998***	(-89.91)	-0.569***	(-105.46)
$\delta\_3M, 6M$	0.169**	(2.80)	0.296**	(2.78)	0.350***	(4.72)
$\delta\_3M, 1Y$	0.232***	(3.78)	0.288*	(2.33)	0.320***	(4.12)
Constant	992188.9***	(1841104.44)	994344.2***	(968824.13)	997785.6***	(2088493.81)
Unsorted	Pre (5)		Crisis (5)		Post (5)	
λ_3M	-0.694***	(-89.95)	-1.032***	(-91.37)	-0.590***	(-107.63)
$\delta\_3M, 6M$	0.0780	(1.24)	0.217*	(2.01)	0.247**	(3.27)
$\delta\_3M, 1Y$	0.159*	(2.50)	0.202	(1.60)	0.246**	(3.12)
Constant	992193.1***	(1770957.30)	994349.1***	(952226.10)	997788.7***	(2054325.85)
Observations	10822		12530		22642	

t statistics in parentheses, * $p < 0.05$, ** $p < 0.01$, *** $p < 0.001$

Table C.5: Robustness (2) 6M Auction

Sorted	Pre (0.05)		Crisis (0.05)		Post (0.05)	
λ_6M	-3.631***	(-85.29)	-4.809***	(-75.46)	-2.835***	(-110.07)
$\delta\_6M, 3M$	0.0348	(0.53)	-0.408***	(-4.06)	0.272***	(5.81)
$\delta\_6M, 1Y$	0.307*	(2.29)	0.569*	(2.13)	0.907***	(5.32)
Constant	985572.7***	(1030903.06)	990016.3***	(536751.43)	995780.1***	(1230965.02)
Sorted	Pre (0.5)		Crisis (0.5)		Post (0.5)	
λ_6M	-3.113***	(-80.97)	-4.328***	(-70.30)	-2.529***	(-103.48)
$\delta\_6M, 3M$	0.0257	(0.43)	-0.374***	(-3.85)	0.231***	(5.22)
$\delta\_6M, 1Y$	0.265*	(2.19)	0.546*	(2.11)	0.774***	(4.79)
Constant	985555.4***	(1141446.47)	989994.8***	(555714.09)	995766.8***	(1297220.30)
Sorted	Pre (5)		Crisis (5)		Post (5)	
λ_6M	-3.631***	(-85.29)	-4.809***	(-75.46)	-2.835***	(-110.07)
$\delta\_6M, 3M$	0.0348	(0.53)	-0.408***	(-4.06)	0.272***	(5.81)
$\delta\_6M, 1Y$	0.307*	(2.29)	0.569*	(2.13)	0.907***	(5.32)
Constant	985572.7***	(1030903.06)	990016.3***	(536751.43)	995780.1***	(1230965.02)
Unsorted	Pre (0.05)		Crisis (0.05)		Post (0.05)	
λ_6M	-2.914***	(-74.17)	-4.043***	(-63.85)	-2.399***	(-96.03)
$\delta\_6M, 3M$	-0.0218	(-0.36)	-0.475***	(-4.75)	0.179***	(3.94)
$\delta\_6M, 1Y$	0.168	(1.36)	0.384	(1.44)	0.694***	(4.20)
Constant	985549.9***	(1117091.97)	989986.1***	(540202.54)	995762.3***	(1268981.08)
Unsorted	Pre (0.5)		Crisis (0.5)		Post (0.5)	
λ_6M	-3.011***	(-75.92)	-4.110***	(-64.62)	-2.443***	(-97.18)
$\delta\_6M, 3M$	-0.0231	(-0.38)	-0.479***	(-4.77)	0.179***	(3.93)
$\delta\_6M, 1Y$	0.163	(1.30)	0.368	(1.38)	0.682***	(4.10)
Constant	985553.4***	(1106620.10)	989989.3***	(537765.34)	995764.5***	(1261419.48)
Unsorted	Pre (5)		Crisis (5)		Post (5)	
λ_6M	-3.373***	(-73.86)	-4.434***	(-65.70)	-2.615***	(-94.85)
$\delta\_6M, 3M$	-0.0884	(-1.25)	-0.575***	(-5.40)	0.153**	(3.05)
$\delta\_6M, 1Y$	0.0485	(0.34)	0.246	(0.87)	0.639***	(3.50)
Constant	985567.5***	(961163.14)	990006.8***	(506913.73)	995774.2***	(1150239.48)
Observations	9072		9221		15974	

t statistics in parentheses, * $p < 0.05$, ** $p < 0.01$, *** $p < 0.001$

Table C.6: Robustness (2) 12M Auction

Sorted	Pre (0.05)		Crisis (0.05)		Post (0.05)	
λ_1Y	-6.127***	(-93.97)	-8.905***	(-81.03)	-4.847***	(-104.09)
$\delta\_1Y,3M$	-0.367***	(-3.61)	-1.742***	(-9.93)	-0.0194	(-0.21)
$\delta\_1Y,6M$	1.164***	(5.43)	3.209***	(7.19)	0.949***	(3.30)
Constant	969544.4***	(666558.72)	977961.4***	(322389.85)	990582.0***	(680925.23)
Sorted	Pre (0.5)		Crisis (0.5)		Post (0.5)	
λ_1Y	-6.834***	(-100.91)	-9.482***	(-84.46)	-5.184***	(-109.77)
$\delta\_1Y,3M$	-0.366***	(-3.47)	-1.748***	(-9.76)	0.0180	(0.19)
$\delta\_1Y,6M$	1.104***	(4.96)	3.090***	(6.77)	0.966***	(3.31)
Constant	969569.1***	(641567.61)	977987.2***	(315592.78)	990597.3***	(671392.12)
Sorted	Pre (5)		Crisis (5)		Post (5)	
λ_1Y	-10.06***	(-77.16)	-12.54***	(-81.72)	-6.898***	(-101.13)
$\delta\_1Y,3M$	-0.340	(-1.68)	-2.146***	(-8.76)	0.182	(1.37)
$\delta\_1Y,6M$	0.320	(0.75)	2.775***	(4.45)	1.535***	(3.64)
Constant	969685.6***	(333426.55)	978127.9***	(230996.06)	990675.7***	(464853.53)
Unsorted	Pre (0.05)		Crisis (0.05)		Post (0.05)	
λ_1Y	-5.914***	(-87.97)	-8.754***	(-78.56)	-4.669***	(-97.69)
$\delta\_1Y,3M$	-0.461***	(-4.40)	-1.805***	(-10.15)	-0.0838	(-0.90)
$\delta\_1Y,6M$	0.962***	(4.35)	3.086***	(6.82)	0.715*	(2.42)
Constant	969540.1***	(646376.42)	977957.8***	(317969.66)	990577.1***	(663474.17)
Unsorted	Pre (0.5)		Crisis (0.5)		Post (0.5)	
λ_1Y	-6.519***	(-91.91)	-9.259***	(-80.81)	-4.896***	(-99.38)
$\delta\_1Y,3M$	-0.509***	(-4.61)	-1.841***	(-10.07)	-0.0984	(-1.02)
$\delta\_1Y,6M$	0.807***	(3.46)	2.910***	(6.25)	0.607*	(1.99)
Constant	969562.8***	(612682.90)	977982.0***	(309204.52)	990589.4***	(643634.43)
Unsorted	Pre (5)		Crisis (5)		Post (5)	
λ_1Y	-8.768***	(-62.68)	-11.59***	(-71.28)	-5.500***	(-72.13)
$\delta\_1Y,3M$	-0.883***	(-4.05)	-2.546***	(-9.81)	-0.460**	(-3.08)
$\delta\_1Y,6M$	-0.935*	(-2.03)	2.004**	(3.03)	-0.0816	(-0.17)
Constant	969659.8***	(310709.69)	978105.4***	(217973.47)	990637.5***	(415792.81)
Observations	12753		10247		19830	

t statistics in parentheses, * $p < 0.05$, ** $p < 0.01$, *** $p < 0.001$

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