Appendix A--The Fundamental Region

The objective is to verify that the fundamental region is as described in section three. This is done via a series of figures.

Figure (A-1) shows that $\dot{q} > 0$ along q=0 for $r > -1/(N-1)$; $\dot{q} < 0$ along q=1 and that $\dot{r} > 0$ for $r < 0$ and sufficiently small in absolute value. Let us verify these. For q=0 from (2-15) and (2-16) describing $\dot{q}$

$$\dot{q} = (b\delta k\beta)(1+(N-1)r)N \qquad \text{(A-1)}$$

which is positive for $r > -1/(N-1)$. Along q=1

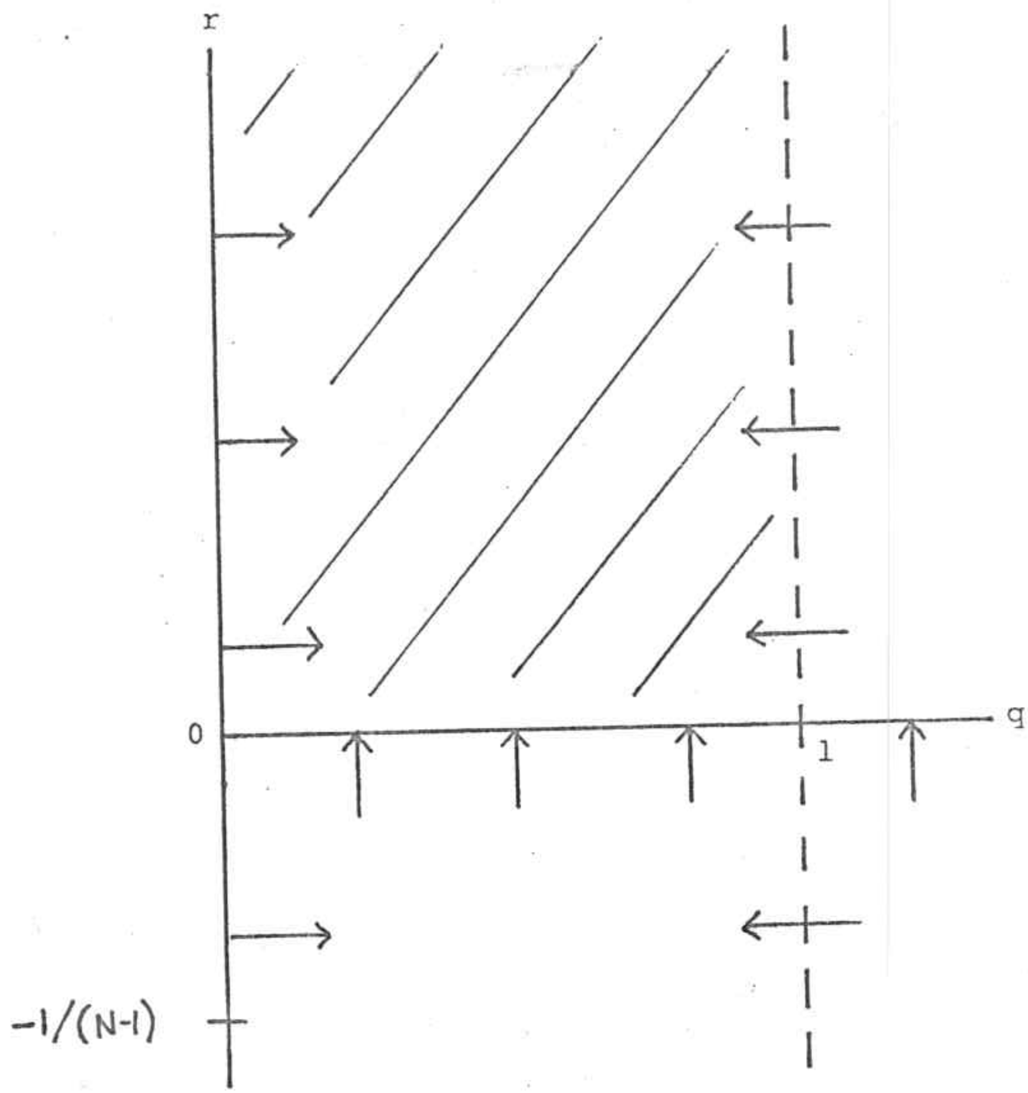
$$\dot{q} = -(b\delta k\beta)(1+(N-1)r)^2 \qquad \text{(A-2)}$$

which is negative. Along r=0 from (2-18) describing $\dot{r}$

$$\dot{r} = b^2\delta\{\delta^2 k[(\pi_j^j)^2 + (\pi_k^j)^2] - m \text{ sgn } r\} \qquad \text{(A-3)}$$

which is strictly positive when $r=-\varepsilon$ since by (2-16) π_j^j and π_k^j don't vanish simultaneously.

Now let us verify that any path starting with r not too negative reaches the shaded region $0 \leq q \leq 1$ $r \geq 0$ in figure (A-1) in finite time. This requires two steps: (1) showing that $r < 0$ but small enough in absolute



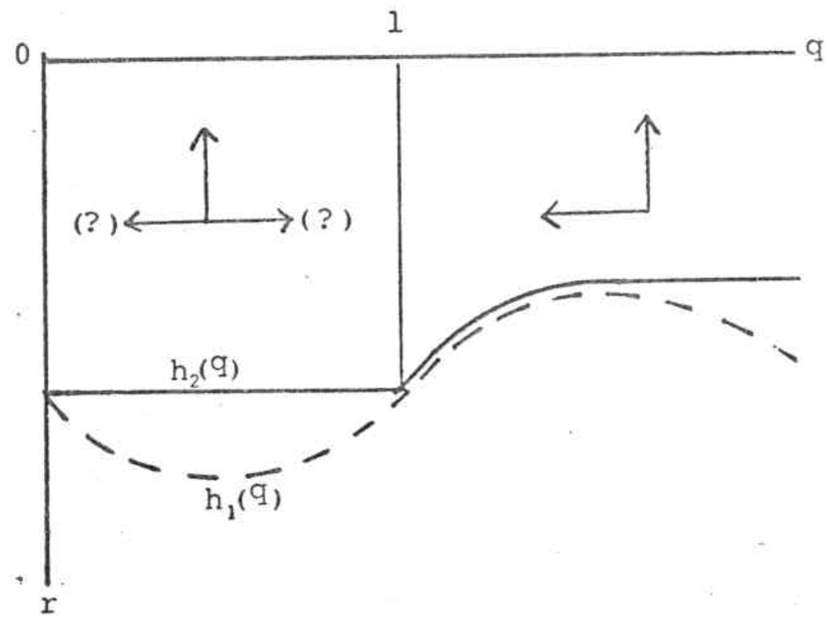
Figure(A-1): Simple Global Features of the Flow value implies $r \geq 0$ is reached in finite time and that (2) the shaded region is reached from any point in the region $q \geq 1$, $r \geq 0$ in finite time. The latter is easy. From (A-2) along $q=1$ $r \geq 0$ $\sup(\dot{q}) = \dot{q}^m < 0$; from (3-2) and (3-5) in the text this implies that for $q \geq 1$ $r \geq 0$ $\sup(\dot{q}) = \dot{q}^m$ also. Thus starting at q in this region it takes no longer than $(q-1)/|\dot{q}^m|$ to reach the shaded region.

Showing $r \geq 0$ reached in finite time from $r < 0$ and small is slightly more difficult. By (A-3) there is a function $h(q)$ such that for $0 > r > h(q)$ $\dot{r}(r,q) > 0$. Since $\dot{r}$ is continuous in q and r for $r < 0$ and since

$$\lim_{\substack{r \to 0 \\ r<0}} \dot{r}(r,q) > 0 \tag{A-4}$$

again by (A-3), we may assume h(q) is a continuous function and that $0 > r > h(q)$ implies $\dot{r}(r,q) > \varepsilon$ for some fixed $\varepsilon > 0$. Since h(q) is continuous we can also assume h(q) constant and greater than $-1/(N-1)$ on $0 \leq q \leq 1$ and strictly increasing for $q > 1$. This is illustrated in figure (A-2). Examining that figure and observing as above that for $q > 1$ $r > -1/(N-1)$ $\dot{q} < 0$ we see that the system once in the region $0 > r > h(q)$ can leave only if r becomes non-negative. But this takes no more time than $1/\varepsilon(N-1)$.

Finally we study the shaded region. First, we show that every path in the shaded region remains bounded. For if not along that path $r \to \infty$ and it must be that



Figure(A-2): The Case $r < 0$

observe that if $h_1(q)$ has $\dot{r}>\varepsilon$ for $0>r>h_1(q)$
then $\dot{r}>\varepsilon$ for $0>r>h_2(q)$ as well

$\dot{r}/\dot{q}$ is unbounded. But as $r \to \infty$ from the equations of motion (2-15) and (2-18) and the profit derivatives (2-16).

$$\frac{\dot{r}}{\dot{q}} \to \left(\frac{b\delta^2}{k\beta N}\right) \frac{r\pi_k^j(N\pi_j^j + (N-2)\pi_k^j)}{r^2(N-1)^2\pi_k^j}$$

$$= \left(\frac{b\delta^2}{k\beta N(N-1)^2}\right) \frac{(N\pi_j^j + (N-2)\pi_k^j)}{r} \to 0 \qquad \text{(A-5)}$$

a contradiction. Next observe that

$$\frac{\partial\dot{r}}{\partial r} = (b^2\delta^3 k E^2/N^2)[(N^2+2N-2)q - N^2]q \qquad \text{(A-6)}$$

from (2-16) and (2-18). It is easy to check that for

$$q^r \equiv \frac{N^2}{N^2 + 2N - 2} > 0 \qquad \text{(A-7)}$$

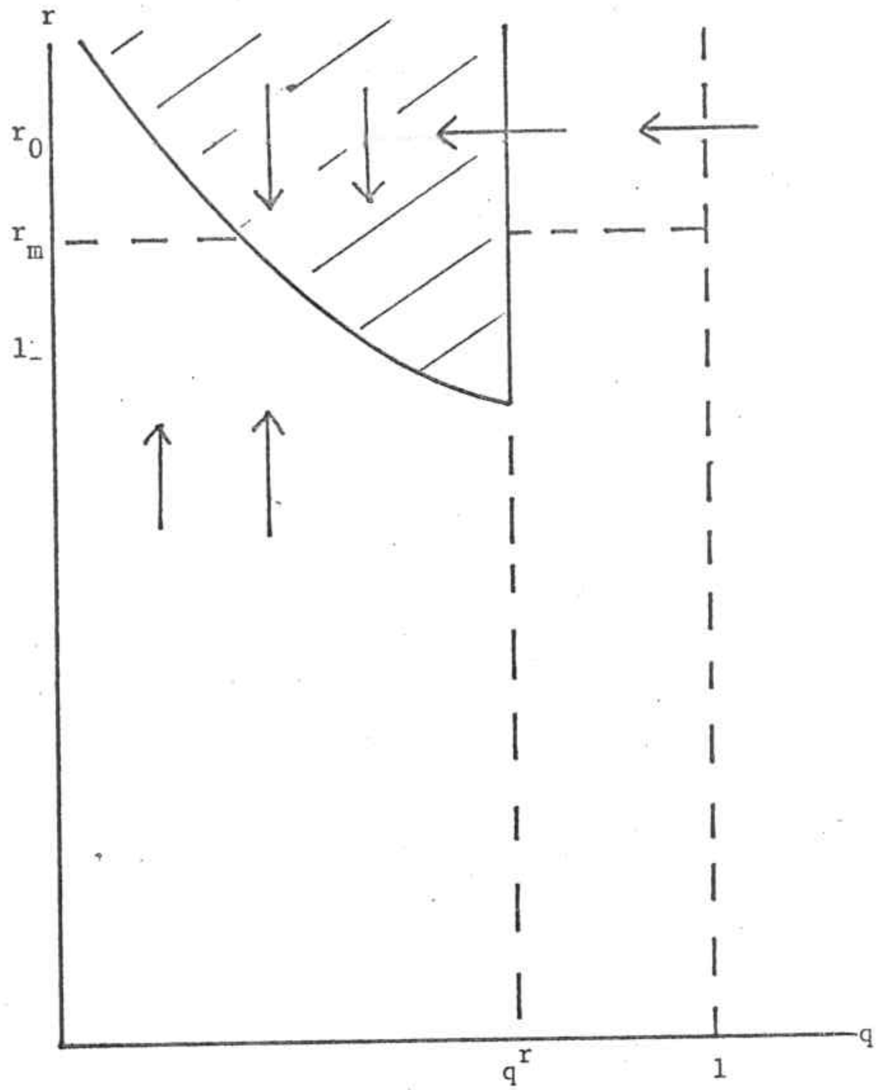
$0 < q < q^r$ implies $\partial\dot{r}/\partial r < 0$. This leads us to figure (A-3). The curve $\dot{r}=0$ from (3-6) is sketched. In the shaded region $\dot{r} < 0$. Note that the shaded region may reach the $q=0$ axis. This doesn't affect the analysis. As indicated in the figure for some $r_m > 1$ and $r \geq r_m$, $q^r \leq q \leq 1$ $\dot{q} < 0$ must hold--this can be verified from figure (3-2) in the text. Also for $0 \leq q < q^r$ and r below the $\dot{r}=0$ curve $\dot{r} > 0$. I next show how to construct the fundamental region.

Consider first the segment $r=r_m$ $q^r \leq q \leq 1$. Every path beginning here is bounded and $\dot{q} < 0$ whenever the path is above r_m and to the right of q^r. This implies that there is a continuous curve C_1 beginning at q=1, $r=r_m$ which meets the shaded region in figure (A-3) and which the flow does not cross from below. Let r_1^m be the r-coordinate where C_1 meets the shaded region. This is shown in figure (A-4).

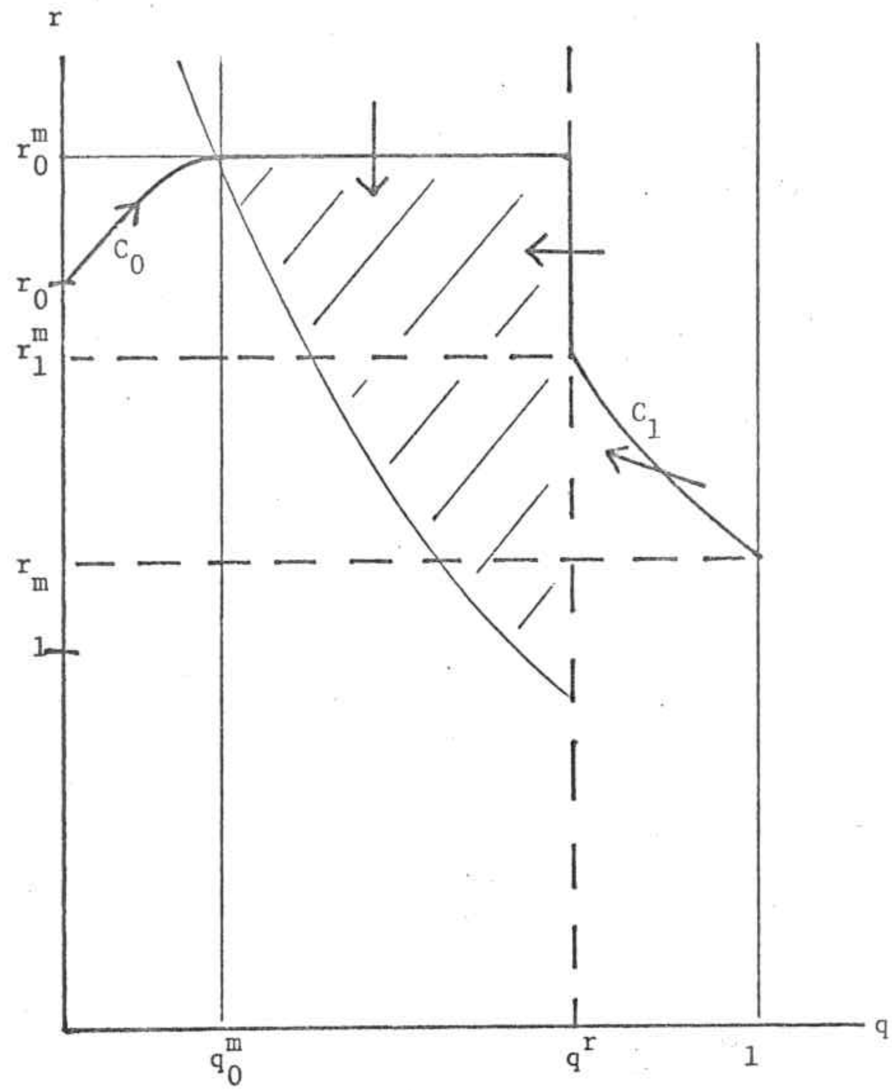
Next consider the path beginning at $r_0 \geq r_1^m$ shown in figure (A-4). Since $\dot{r} > 0$ and the path is bounded above, this curve meets the shaded region at some ordinate $r_0^m \geq r_0 \geq r_1^m$. Since this is an integral path it is not crossed by the flow. Comparing figures (A-3) and (A-4) we see that the piecewise continuous curve made up of C_0, the segment $r=r_0^m$, $q_1^m \leq q \leq q^r$, the segment $q=q^r$ $r_0^m \leq r \leq r_1^m$ and the curve C_1 connects q=0 and q=1 and is crossed only from above. This defines the upper boundary of the fundamental region shown in figure (3-1).

It remains to show that paths beginning above the fundamental region F reach it in finite time. To show this we observe that the ω-limit set of such a path is bounded, and therefore a compact non-empty set W. We now apply some results on planar systems from Hirsch and Smale [8] chapter 11.

We may as well assume $W \cap F = \phi$, otherwise the path reaches F in finite time since no limit point lies on the boundary of F. By the Poincaré-Bendixon theorem either W



Figure(A-3): Behavior of the Reaction Coefficient



Figure(A-4): Bounding the Fundamental Region contains a steady state or it is a closed orbit with a steady state in its interior: either case implies that the region above F contains a steady state. However, every steady state has $r \leq 1 < r_m$ by the results of section three and this contradiction establishes the required result.

Appendix B--Instability with Negative Response

The objective is to show that any steady state with $r < 0$ is unstable.

The starting point of the analysis is the equation of motion for $\dot{q}$. From (2-15) and (2-16) this is

$$\dot{q} = (b\delta k \beta)(1+(N-1)r)(N-(1+N)q-(N-1)rq) \tag{B-1}$$

Inspection shows $\dot{q}=0$ either when $r = -1/(N-1)$ or along the curve given in (3-2) as

$$r = \frac{N - (1+N)q}{(N-1)q} \tag{B-2}$$

Since this curve strictly decreases and $r \to -(1+N)/(N-1)$ as $q \to \infty$, we may assume $0 > r > -(1+N)/(N-1)$. There are three cases $0 > r > -1/(N-1)$; $r = -1/(N-1)$ and $-1/(N-1) > r > -(1+N)/(N-1)$.

Case 1: $0 > r > -1/(N-1)$

Then any steady state lies along (B-2) implying $q^N < q < 1$. Using (2-16) and (2-18) to find the motion of r, and substituting in (B-2) shows the steady state is at

$$q^1 = \frac{3}{4} \pm \sqrt{\frac{1}{16} + M} \tag{B-3}$$

where M is as in (3-9). Since $M > 0$ either

$$q > 1 \tag{B-4}$$

or

$$q < (1/2) = q^M < q^N \tag{B-5}$$

So in either case there is a contradiction. Thus in case (1) there can be no steady state.

<u>Case 2</u>: $r = -1/(N-1)$

Using the given value of r and the equation of motion for r from (2-16) and (2-18) shows that at a steady state

$$[N^2-2]q^2 - [2N^2-N-2]q + N(N-1) = -\bar{M} \tag{B-6}$$

where $\bar{M} > 0$ is a constant. Inspection of the polynomial in (B-6) shows that the steady state value of q must then lie in the interval

$$1 > q > \frac{N(N-1)}{N^2 - 2} > \frac{N^2}{N^2 + 2N - 2} = q^r \tag{B-7}$$

where q^r was defined in appendix (A) in (A-7). But the analysis of that section, and (A-6) in particular, showed that $q > q^r$ implies $\partial\dot{r}/\partial r > 0$. On the other hand from section three equation (3-5) $\partial\dot{q}/\partial q = 0$. Thus $\partial\dot{r}/\partial r + \partial\dot{q}/\partial q > 0$ implying instability.

<u>Case 3:</u> $-1/(N-1) > r > (1+N)/(N-1)$

The steady state then lies along (B-2) implying $q > 1$. Equation (3-5) then shows $\partial\dot{q}/\partial q > 0$ while (A-6) shows since $q > q^r$ $\partial\dot{R}/\partial R > 0$. Thus $\partial\dot{q}/\partial q + \partial\dot{R}/\partial R > 0$ contradicting stability.

Appendix C--Stability with Positive Response

The objective is to verify that the steady state at q^S is stable, that at q^U is unstable. Observe that these steady states occur at the intersection of the curves $\dot{q}=0$ from (3-2)

$$r = \frac{N - (1+N)q}{(N-1)q} \tag{C-1}$$

and $\dot{r}=0$ from (3-6)

$$r = \frac{(mN^2/E^2\delta^2 k) - [(N^2+2N+2)q^2 - 2N(1+N)q + N^2]}{[(N^2+2N-2)q - N^2]q} \tag{C-2}$$

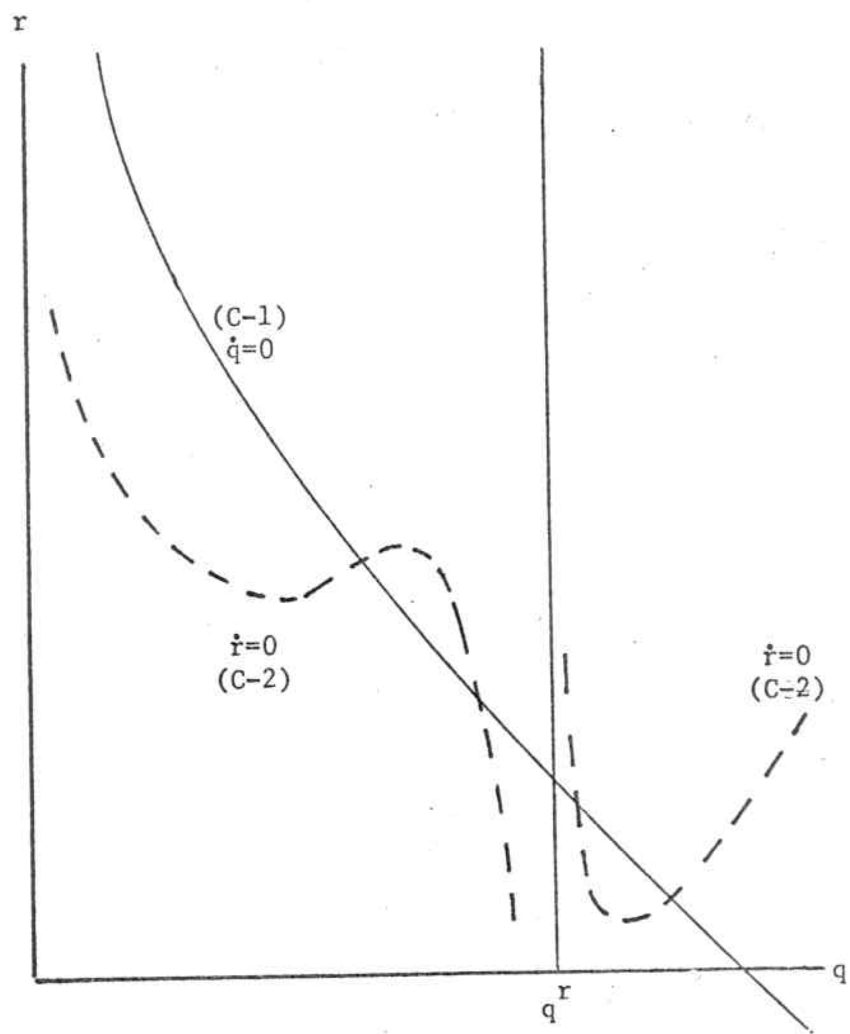
It is straightforward to verify that if these curves intersect at all and $q^S \neq q^U$ they intersect exactly twice at q^S and q^U. By definition q^S is the first intersection of these curves, q^U the second. From (3-5) at a steady state $\partial\dot{q}/\partial q < 0$ while $\partial\dot{r}/\partial r < 0$ if $q < q^r$ $\partial\dot{r}/\partial r > 0$ if $q > q^r$. Observe that the curve (C-2) has a pole at q^r and is continuous on $0 < q < q^r$ and on $q^r < q$. This information suffices, using some results from Levine [12] section (2) to determine the stability of the steady states: for $q < q^r$ a steady state is stable if and only if the curve (C-2) intersects the curve (C-1) from below; for

$q > q^r$ a steady state is unstable if the curve (C-2) intersects (C-1) from below, stable if from above and in addition $\partial\dot{q}/\partial q + \partial\dot{r}\partial r < 0$. However, the later condition always holds for b small enough--$\partial\dot{q}/\partial q < 0$ and is of order b while $\partial\dot{r}/\partial r$ is only of order b^2.

Consider first $q < q^r$. As $q \to 0$ (C-1) approaches $(N/(N-1))(1/q)$ while (C-2) approaches $\{[(mN^2/E^2\delta^2 R) - N^2]/N^2\}(1/q)$ from which it is seen as $q \to 0$ (C-1) lies above (C-2). Thus if $q^S < q^r$, since the first intersection of curves must be with (C-2) hitting (C-1) from below, q^S is stable. If $q^U < q^r$ it is at the second intersection which has (C-2) hitting (C-1) from above (since both curves are continuous on $0 < q < q^r$) and is unstable.

Taking the other case $q > q^r$ from (C-2) it is clear that as $q \to \infty$ (C-2) goes to $+\infty$, while from (C-1) the $\dot{q}=0$ curve becomes negative. Thus if $q^U > q^r$ it is at the second intersection with (C-2) hitting (C-1) from below and is unstable; if $q^S > q^r$ it is at the first intersection and is stable.

This line of reasoning is illustrated in figure (C-1).



Figure(C-1): Intersections of $\dot{q}=0$ and $\dot{r}=0$

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Notes

(1) See for example Laitner [9].

(2) Supergames are a concept due to Friedman [2].

(3) For a discussion of this see Friedman [3]. Marschak and Selten [13, 14] or Radner [15]. The relevant equilibrium concept is that of perfect equilibrium found in Selten [18]. Similar results hold for static conjectural equilibria discussed by Hahn [6] and Seade [17].

(4) The relevant environment is a continuous time model with discounting and adjustment costs.

(5) Non-identical firms and non-quadratic profit functions are examined in Levine [11].

(6) This differs from the formulation of Marschak and Selten [14] in that firms respond the same way to both increases and decreases in output by opponents. In the differentiable framework here there is no advantage to kinked response: the optimal response to punish opponents for cheating and the optimal response to reward them for colluding are the same.

(7) Reactions apply only to future changes in output and do not apply retroactively to past deviations by rival firms. This distinguishes the present model from the formulation in Guttman [5].

(8) More general technologies are examined in Levine [11]. If firms face a capacity constraint I assume that it is sufficiently large that it is not binding in competitive equilibrium.

(9) One insignificant difference between the two approaches is that when adjustment costs are explicitly introduced $\hat{A}^j$ must include an estimate of the present value of future adjustment costs. Fortunately Levine [10] shows that in the present case the only effect of this is to introduce some irrelevant constants into the adjustment equation.

(10) A mathematical technicality of no economic import is that $C^j(R^j)$ is not differentiable when $R^j_k=0$. This is ignored.

(11) With this simplification the model is formally and conceptually similar to that of Guttman [5]. I am grateful to Dr. Guttman for making available unpublished research conducted jointly with Michael Miller along lines similar to those here.

(12) See Hirsch and Smale [9] chapter 16.

(13) See Levine [11] for results with asymmetric initial conditions.

(14) For elementary catastrophe theory see Zeeman [21] especially essays one and ten.

(15) See Scherer [16], pp. 158-164 for example.